The Perturbation Lemma for Linear Factorisations

Rohan Hitchcock

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1 The Perturbation Lemma

The Perturbation Lemma is a powerful result with broad applications, which are summarised in [Cra04]. Roughly speaking, it is a tool which allows us to modify the differentials of homotopy equivalent complexes or linear factorisations without disturbing the equivalence, subject to certain conditions on the nature of the homotopy equivalence. The result is usually stated for chain complexes in an abelian category (see [BL91; Cra04]) however it is readily adapted to the setting of linear factorisations. The definitions and results relating to linear factorisations in this section are all analogues to similar statements for chain complexes in an abelian category. These analogous statements are summarised in Appendix A.

Let S be a commutative ring and R a S-algebra.

Definition 1.1. Let (L, d_L) and (M, d_M) be linear factorisations of $f \in R$. A *deformation retract* over S between (L, d_L) and (M, d_M) consists of S-linear maps

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h \tag{$*$}$$

where $pi = 1$, and ip is homotopic to the identity via h (so $ip - 1 = hd_M + d_L h$). The deformation retract is called *strong* if $hi = 0$, $ph = 0$ and $h^2 = 0$.

A (strong) deformation retract is a special type of homotopy equivalence of linear factorisations. In the setting of the above definition, a *perturbation* of $(*)$ is an odd R-linear map $\delta : M \to M$ where $(d_M + \delta)^2 = g \cdot 1$, where possibly $f \neq g$. The perturbation δ is called *small* if $(1 - \delta h)$ is invertible. This is frequently checked in the following way.

Lemma 1.2. *A perturbation δ is small if and only if $(\delta h)^n = 0$ for sufficiently large n.*

Proof. If $(\delta h)^n = 0$ then the inverse of $(1 - \delta h)$ is $\sum_{k=0}^{n} (\delta h)^k$. The converse is explained in [Cra04, Remark 2.3.iii].

TODO: add proof of converse □

Remark. A (strong) deformation retract of complexes of S-modules is defined in the obvious way (Definition A.3), and as are (small) perturbations thereof. When regarding a complex as a linear factorisation of zero, a (strong) deformation retract of complexes is exactly the same as a (strong) deformation retract of linear factorisations.

We now move to the statement of the Perturbation Lemma. Let $\varphi : S \to R$ be the morphism of rings associated to the S-algebra structure on R and note that if $b : M \to N$ is an S-linear map of R-modules and $r \in \varphi(S)$ then $b(rm) = rb(m)$ for all $m \in M$.

Theorem 1.3 (Perturbation Lemma for linear factorisations)**.** *Suppose we have a strong deformation retract over S of linear factorisations of $f \in \varphi(S)$*

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h$$

and let δ be a small perturbation such that $(d_M + \delta)^2 = g \cdot 1$ and $g \in \varphi(S)$. Then the perturbed data

$$(L, d'_L) \underset{i'}{\overset{p'}{\leftrightarrows}} (M, d_M + \delta), \qquad h'$$

is a deformation retract over S of linear factorisations of g, where $d'_L = d_L + pai$, $i' = i + hai$, $p' = p + pah$, $h' = h + hah$ and $a = (1 - \delta h)^{-1}\delta$.

Proof. The proof of this theorem closely follows the proof in [Cra04, Section 2.4] of the analogous statement for complexes. We begin by proving the following statements:

(1) $\delta ha = ah\delta = a - \delta$,

(2) $(1 - \delta h)^{-1} = 1 + ah$ and $(1 - h\delta)^{-1} = 1 + ha$,

(3) $aipa + ad_M + d_M a = (g \cdot 1 - f \cdot 1)(1 + ah + ha)$.

For (1), note by definition of a we have $(1 - \delta a) = \delta$, proving $a - \delta = \delta ha$. For the other equality, we can write $\delta h \delta = \delta - (1 - \delta h)\delta$ and multiply on the left by $(1 - \delta h)^{-1}$ to get $ah\delta = a - \delta$. Statement (2) is proved by observing

$$\begin{aligned}(1 + ah)(1 - \delta h) &= 1 + ah - \delta h - ah\delta h \\ &= 1 + ah - \delta h - (a - \delta)h \\ &= 1\end{aligned}$$

and similarly that $(1 + ha)(1 - h\delta) = 1$, $(1 - h\delta)(1 + ha) = 1$ and $(1 + ha)(1 - h\delta) = 1$. For (3) we compute directly. Using (1) and (2) above, and the fact that $h^2 = 0$ we have

$$\begin{aligned}ad_M + d_M + aipa &= ad_M + d_M a + a(1 + d_M h + hd_M)a \\ &= ad_M(1 + ha) + (1 + ah)d_M a + a^2 \\ &= ad_M(1 - h\delta)^{-1} + (1 - \delta h)^{-1} d_M a + a^2 \\ &= (1 - \delta h)^{-1}\big[(1 - \delta h)ad_M + d_M a(1 - h\delta) \\ &\qquad + (1 - \delta h)a^2(1 - h\delta)\big](1 - h\delta)^{-1} \\ &= (1 + ah)\big[(a - \delta ha)d_M + d_M(a - ah\delta) \\ &\qquad + (a - \delta ha)(a - ah\delta)\big](1 + ha) \\ &= (1 + ah)\big[(a - a + \delta)d_M + d_M(a - a + \delta) \\ &\qquad + (a - a + \delta)(a - a + \delta)\big](1 + ha) \\ &= (1 + ah)\big[\delta d_M + d_M\delta + \delta^2\big](1 + ha) \\ &= (1 + ah)\big[(d_M + \delta d_M)^2 - d_M^2\big](1 + ha) \\ &= (g \cdot 1 - f \cdot 1)(1 + ah)(1 + ha) \\ &= (g \cdot 1 - f \cdot 1)(1 + ah + ha)\end{aligned}$$

<u>(L, d'_L) is a linear factorisation of g.</u> We need to show that ${d'_L}^2 = g \cdot 1$. We have

$$
\begin{aligned}
{d'_L}^2 &= (d_L + pai)^2 \\
&= f \cdot 1 + d_L pai + paid_L + paipai \\
&= f \cdot 1 + d_L pai + paid_L + p\left((g \cdot 1 - f \cdot 1)(1 + ah + ha) - ad_M - d_M a\right) i \\
&= g \cdot 1 + d_L pai + paid_L + p\left((g \cdot 1 - f \cdot 1)(ah + ha) - ad_M - d_M a\right) i \\
&= g \cdot 1 + pa\big(id_L + (g \cdot 1 - f \cdot 1)hi - d_M i\big) + \big(d_L p + (g \cdot 1 - f \cdot 1)ph - pd_M\big)ai \\
&= g \cdot 1
\end{aligned}
$$

where we use that $pi = 1$, $pd_M = d_L p$, $id_L = d_M i$, $hi = 0$, $ph = 0$, $h^2 = 0$, and equation (3) above.

<u>i' is a morphism.</u> We need to show that $i'd'_L = (d_M + \delta)i'$. We have

$$
\begin{aligned}
i'd'_L - (d_M + \delta)i' &= (i + hai)(d_L + pai) - (d_M + \delta)(i + hai) \\
&= id_L + haid_L + ipai + haipai - d_M i - \delta i - d_M hai - \delta hai \\
&= id_L + haid_L + ipai + h\big((g \cdot 1 - f \cdot 1)(1 + ah + ha) - ad_M - d_M a\big)i \\
&\qquad - d_M i - \delta i - d_M hai - (a - \delta)i \\
&= haid_L + ipai - h(ad_M + d_M a)i - d_M hai - ai \\
&= ha(id_L - d_M i) + (ip - hd_M - d_M h - 1)ai \\
&= 0
\end{aligned}
$$

where we use (1), (3), $id_L = d_M i$, $hi = 0$, $h^2 = 0$, $ip - 1 = hd_M + d_M h$ and $id_L = d_M i$.

<u>p' is a morphism.</u> We need to show that $d'_L p' = p'(d_M + \delta)$. We have

$$
\begin{aligned}
d'_L p' - p'(d_M + \delta) &= (d_L + pai)(p + pah) - (p + pah)(d_M + \delta) \\
&= d_L p + paip + d_L pah + paipah - pd_M - pahd_M - pah\delta - p\delta \\
&= d_L p + paip + d_L pah + p\big((g \cdot 1 - f \cdot 1)(1 + ah + ha) - ad_M - d_M a\big)h \\
&\qquad - pd_M - pahd_M - p(a - \delta) - p\delta \\
&= paip + d_L pah - p(ad_M + d_M a)h - pahd_M - pa \\
&= pa(ip - d_M h - hd_M - 1) + (d_L p - pd_M)ah \\
&= 0
\end{aligned}
$$

where we use (1), (3), $h^2 = 0$, $ph = 0$, $d_L p = pd_M$ and $ip - 1 = d_M h + hd_M$.

<u>$p'i' = 1$.</u> This is straightforward:

$$
p'i' = (p + pah)(i + hai) = 1
$$

since $ph = 0$, $hi = 0$ and $h^2 = 0$.

<u>h' is a homotopy from $i'p'$ to 1.</u> We need to show $i'p' - 1 = h'(d_M + \delta) + (d_M + \delta)h'$.

Writing $d'_M = d_M + \delta$ we have

$$\begin{aligned}
1 + h'd'_M + d'_M h' - i'p' &= 1 + (h + hah)(d_M + \delta) + (d_M + \delta)(h + hah) \\
&\qquad - (i + hai)(p + pah) \\
&= 1 + hd_M + hahd_M + h\delta + hah\delta + d_M h + d_M hah + \delta h + \delta hah \\
&\qquad - ip - ipah - haip - haipah \\
&= hahd_M + h\delta + hah\delta + d_M hah + \delta h + \delta hah \\
&\qquad - ipah - haip - haipah \\
&= hahd_M + h\delta + h(a - \delta) + d_M hah + \delta h + (a - \delta)h - ipah \\
&\qquad - haip - h\big((g \cdot 1 - f \cdot 1)(1 + ah + ha) - ad_M - d_M a\big)h \\
&= hahd_M + ha + d_M hah + ah - ipah - haip + h\big(ad_M + d_M a\big)h \\
&= ha(hd_M + 1 - ip + d_M h) + (d_M h + 1 - ip + hd_M)ah \\
&= 0
\end{aligned}$$

where we use (1), (3), $ip - 1 = hd_M + d_M h$ and $h^2 = 0$.

This has shown that the maps i', p', d'_L, d'_M and h' form a deformation retract. It is clearly also a strong deformation retract, and so this proves the claim. □

Remark. One can show that we can replace the condition that the initial deformation retract in Theorem 1.3 be strong with the following conditions on the deformation retract and small perturbation:

(1) $p\delta = 0$ and $ph = 0$,

(2) $(d_M + \delta)^2 = d_M^2$.

2 Perturbation in $\mathcal{LG}_k$

Lemma 2.1 (Proposition 6.1 [DM13])**.** *Let S be a ring and R an S-algebra. Let (L, d_L) and (M, d_M) be linear factorisations of $f \in \varphi(S)$ and suppose we have a strong deformation retract over S*

$$(L, d_L) \underset{\sigma}{\overset{\pi}{\leftrightarrows}} (M, d_M), \quad h \qquad (*)$$

Then for any linear factorisation (Z, d_Z) of $g \in R$ where $f + g \in \varphi(S)$ there exists a deformation retract over S

$$(L \otimes_R Z, d_L \otimes 1 + 1 \otimes d_Z) \leftrightarrows (M \otimes_R Z, d_M \otimes 1 + 1 \otimes d_Z), \quad h'$$

Proof. Tensoring the modules in $(*)$ by Z we obtain

$$(L \otimes_R Z, d_L \otimes 1) \underset{\pi\otimes 1}{\overset{\sigma\otimes 1}{\leftrightarrows}} (M \otimes_R Z, d_M \otimes 1), \quad h \otimes 1 \ .$$

which is also a strong deformation retract. Finally note that $1 \otimes d_Z$ is a small perturbation since $(1 - h \otimes d_Z)^{-1} = (1 + h \otimes d_Z)$ and so by the Perturbation Lemma (Theorem 1.3) we obtain a strong deformation retract over S:

$$(L \otimes_R Z, d_L \otimes 1 + 1 \otimes d_Z) \leftrightarrows (M \otimes_R Z, d_M \otimes 1 + 1 \otimes d_Z), \quad h'$$

□

Lemma 2.1 provides us with a tool to modify existing strong deformation retracts; the next result provides us with a source.

Lemma 2.2. *Let (P, d) be a bounded-to-the-right chain complex of projective objects in an abelian category. Suppose that (P, d) is exact except at degree zero and that $H_0(P)$ is also projective. Then we have a strong deformation retract*

$$(H(P), 0) \leftrightarrows (P, d), \quad h$$

of chain complexes, where $(H(P), 0)$ is the homology of P with zero differentials.

Proof. Note that $H_0(P) = P_0 / \operatorname{im}(d_1)$ and $H_i(P) = 0$ for $i \neq 0$. A chain map $p : (P, d) \to (H_0(P), 0)$ is obtained by considering the quotient morphism $p_0 : P_0 \to H_0(P)$. A chain map $i : (H_0(P), 0) \to (P, d)$ is obtained from the lifting property of projective objects. Since $H_0(P)$ is assumed to be projective and the quotient map p_0 is an epimorphism we obtain a map $i_0 : H_0(P) \to P_0$ such that

$$\begin{array}{ccc} P_0 & \xrightarrow{p_0} & H_0(P) \\ & \nwarrow^{i_0} & \| \\ & & H_0(P) \end{array}$$

commutes. Hence we have $pi = 1$. We now construct a homotopy $h : 1 \cong ip$.

We construct the maps $h_n : P_n \to P_{n+1}$ by induction on n. For h_0, we are in the following situation

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle 1 - i_0 p_0} & & \\ \cdots & \longrightarrow & P_1 & \xrightarrow{d_1} & P_0 & \longrightarrow & 0 \end{array}$$

Note that $p_0(1 - i_0p_0) = 0$ and so $(1 - i_0p_0)$ factors through $\ker(p_0)$. By definition of p_0 we also have that $d_1 : P_1 \to \ker(p_0)$ is an epimorphism. Then we can apply the lifting property of projective objects to obtain $h_0 : P_0 \to P_1$ such that

$$\begin{array}{ccc} & & P_0 \\ & \swarrow^{h_0} & \downarrow{\scriptstyle 1-i_0p_0} \\ P_1 & \xrightarrow[d_1]{} & \ker(p_0) \subseteq P_0 \end{array}$$

commutes. Now let $n \geq 0$ and suppose we have constructed h_k for $k < n$. Then we are in the situation

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & P_{n+1} & \xrightarrow{d_{n+1}} & P_n & \xrightarrow{d_n} & P_{n-1} & \xrightarrow{d_{n-1}} & P_{n-2} & \longrightarrow & \cdots \\ & & & & \downarrow{\scriptstyle 1} & \swarrow_{h_{n-1}} & \downarrow{\scriptstyle 1} & \swarrow_{h_{n-2}} & \downarrow{\scriptstyle 1} & \swarrow & \\ \cdots & \longrightarrow & P_{n+1} & \xrightarrow[d_{n+1}]{} & P_n & \xrightarrow[d_n]{} & P_{n-1} & \xrightarrow[d_{n-1}]{} & P_{n-2} & \longrightarrow & \cdots \end{array}$$

where we have $1 = d_nh_{n-1} + h_{n-2}d_{n-1}$. Note that if $n = 1$ then $P_{-1} = 0$, $h_{-1} = 0$ and $d_0 = 0$. We now aim to construct $h_n : P_n \to P_{n+1}$. Note that $d_n(1 - h_{n-1}d_n) = d_n - (1 - h_{n-2}d_{n-1})d_n = 0$ so $(1 - h_{n-1}d_n)$ factors through $\ker(d_n)$. Since (P, d) is assumed to be exact in degree n the map $d_{n+1} : P_{n+1} \to \ker(d_n)$ is an epimorphism. Then, using the lifting property of projective objects we have $h_n : P_n \to P_{n+1}$ such that

$$\begin{array}{ccc} & & P_n \\ & \swarrow^{h_n} & \downarrow{\scriptstyle 1-h_{n-1}d_n} \\ P_{n+1} & \xrightarrow[d_{n+1}]{} & \ker(p) \subseteq P_0 \end{array}$$

commutes, or in other words $1 = d_{n+1}h_n + h_{n-1}d_n$. The maps i, p and h form the desired deformation retract, which by Lemma A.4 can be upgraded to a strong deformation retract. □

In defining $\mathcal{LG}_k$, we will always apply Lemma 2.2 in the case that (P, d) is the Koszul complex of some Koszul-regular sequence. The definition of a potential (Definition **??**) is arranged to ensure the conditions of Lemma 2.2 are satisfied. For example, consider the following 1-morphisms $(X, d_X) : (k[x], U) \to (k[y], V)$ and $(Y, d_Y) : (k[y], V) \to (k[z], W)$ in $\mathcal{LG}_k$. When showing their composition is well defined we will apply Lemma 2.2 when (P, d) is the Koszul complex of the sequence of partial derivatives $\partial_{y_1}V, \cdots, \partial_{y_n}V$ considered as elements of $k[x, y, z]$. By assumption the sequence of partial derivatives is Koszul-regular and the Jacobi ring $k[y]/(\partial_{y_1}V, \cdots, \partial_{y_n}V)$ is free over k. The latter results in the degree zero homology of the Koszul complex being free over $k[x, z]$, hence projective.

A Perturbation of complexes in an abelian category

Let $\mathcal{A}$ be an abelian category. In this setting, the Perturbation Lemma applies somewhat more generally.

Definition A.1. A *homotopy equivalence datum* of complexes (L, d_L) and (M, d_M) in $\mathcal{A}$ consists of the following:

(1) A pair of complexes with quasi-isomorphisms $i : L \leftrightarrows M : p$.

(2) A homotopy $h : ip \cong 1$.

This data is usually presented like so:

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h \tag{$*$}$$

Note that a homotopy equivalence datum between complexes is not necessarily a homotopy equivalence of complexes; it is a weaker concept. It is, however, an isomorphism of complexes in the derived category of $\mathcal{A}$. Since we have no notion of 'homology' for linear factorisations — and hence no notion of a 'quasi-isomorphism' — the notion of a homotopy equivalence datum does not make sense for linear factorisations.

In the setting of the definition above, a *perturbation* of the homotopy equivalence datum $(*)$ is a map $\delta : M \to M[1]$ such that $(m + \delta)^2 = 0$. The perturbation δ is called *small* if $(1 - \delta h)$ is invertible. The Perturbation Lemma concerns a small perturbation of a homotopy equivalence datum.

Theorem A.2 (Perturbation Lemma)**.** *Let*

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h$$

be a homotopy equivalence datum in $\mathcal{A}$ and δ a small perturbation. Then the perturbed data

$$(L, d_L') \underset{i'}{\overset{p'}{\leftrightarrows}} (M, d_M + \delta), \qquad h'$$

is also a homotopy equivalence datum, where $d_L' = d_L + pai$, $i' = i + hai$, $p' = p + pah$, $h = h + hah$ and $a = (1 - \delta h)^{-1}\delta$.

Proof. See [Cra04] or the proof of Theorem 1.3. □

Definition A.3. A *deformation retract of complexes* in $\mathcal{A}$ is a homotopy equivalence datum

$$(L, d_L) \underset{i}{\overset{p}{\leftrightarrows}} (M, d_M), \qquad h$$

where we also have $pi = 1$. A DR is called *strong* if $hi = 0$, $ph = 0$ and $h^2 = 0$.

Clearly a deformation retract is also a homotopy equivalence datum. The terminology "strong deformation retract" comes from [BL91]; in [Cra04] this is called "special deformation retract". In [Cra04, Remark 2.3.i] it is noted that any deformation retract can be transformed into a strong deformation retract.

Lemma A.4. *If we have a deformation retract, then the morphisms can be modified to produce a strong deformation retract between the same pair of complexes.*

Corollary A.5 (to Perturbation Lemma)**.** *Any small perturbation of a strong deformation retract (in the sense of Theorem A.2) is a strong deformation retract.*

References

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