Matrix Factorisations

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1 Definitions

Let S be a commutative ring and $f \in S$. The example to keep in mind is the case when S is a polynomial ring over a commutative ring k, possibly in more than one variable.

The most concrete way of thinking about matrix factorisations of f is as a pair of $n \times n$ square matrices (P, Q) with entries in S which satisfy

$$PQ = QP = f \cdot I$$

where I is the $n \times n$ identity matrix. For example if $S = k[x, y]$ and $f = x^2 - y^2$ we have the following matrix factorisation of f:

$$\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} x & -y \\ -y & x \end{pmatrix} = \begin{pmatrix} x & -y \\ -y & x \end{pmatrix} \begin{pmatrix} x & y \\ y & x \end{pmatrix} = \begin{pmatrix} x^2 - y^2 & 0 \\ 0 & x^2 - y^2 \end{pmatrix}$$

By choosing n generators for $S^{\oplus n}$ we can associate to P and Q morphisms $p, q : S^{\oplus n} \to S^{\oplus n}$ respectively. With this view, the data of a matrix factorisation can be expressed like so:

$$\underset{0}{S^{\oplus n}} \xrightarrow{p} \underset{1}{S^{\oplus n}} \xrightarrow{q} \underset{0}{S^{\oplus n}}$$

This can be viewed as a $\mathbb{Z}_2$-graded S-module $X = X_0 \oplus X_1$, where $X_0 = S^{\oplus n}$ and $X_1 = S^{\oplus n}$, together with an odd endomorphism $d_X = \left(\begin{smallmatrix} 0 & q \\ p & 0 \end{smallmatrix}\right)$ such that $d_X^2 = f \cdot 1_X$. This grading is also indicated on the diagram above. There are some immediate generalisations:

- What if we allow matrix factorisations of 'infinite rank'? e.g. $X_i = S^{\oplus \mathbb{Z}}$.
- More generally, what if the X_i are not required to be free S-modules?

Definition 1. A *linear factorisation of* $f \in S$ is a pair (X, d_X) where X is a $\mathbb{Z}_2$-graded S-module and $d_X : X \to X$ is an odd endomorphism such that $d_X^2 = f \cdot 1_X$. A linear factorisation (X, d_X) is a *matrix factorisation* if X is a free S-module, and a matrix factorisation is *finite rank* if the module is finitely generated.

Recall that a complex of S-modules is a $\mathbb{Z}$-graded S-module C equipped with a degree ± 1 endomorphism $d_C : C \to C$ which squares to zero. A linear factorisation can be viewed as a $\mathbb{Z}_2$-graded cousin of a complex of S-modules, although in many respects one which is less interesting; there is no notion of (co)homology for linear factorisations. Nevertheless many concepts from the world of complexes readily extend to linear factorisations.

Definition 2. A *morphism of linear factorisations* $\alpha : (X, d_X) \to (Y, d_Y)$ is a degree zero S-linear map which commutes with the differential, meaning that both squares in the diagram

$$\begin{array}{ccccc} X_0 & \longrightarrow & X_1 & \longrightarrow & X_0 \\ \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_0} \\ Y_0 & \longrightarrow & Y_1 & \longrightarrow & Y_0 \end{array}$$

commute.

2 The geometric content of matrix factorisations

As above let S be a commutative ring, $f \in S$, and set $R = S/(f)$. In this section we will explain how to associate a geometric object to a finite rank matrix factorisation (X, d) of f in the case that f is not a zero divisor. When $S = k[x_1, \cdots, x_m]$ this geometric object will be a part of the *algebraic set* associated to the ring R. That is:

$$\mathbf{V}(f) = \{a \in k^n \mid f(a) = 0\}.$$

Note that in the case of $k = \mathbb{C}$ or $k = \mathbb{R}$ the condition that f is not a zero divisor just means $f \neq 0$. Let (X, d) be a finite rank matrix factorisation of f, where $X = X_0 \oplus X_1$ and $d = \left(\begin{smallmatrix} 0 & d_1 \\ d_0 & 0 \end{smallmatrix}\right)$.

Matrix factorisations to R-modules

The first step is to associate to (X, d) a particular type of R-module. Since $d_1 d_0 = f \cdot 1_{X_1}$ and f is not a zero divisor d_0 is a monomorphism (the same can also be said of d_1). Setting $M = \operatorname{coker}(d_0) = X_1 / \operatorname{im}(d_0)$ we obtain the short exact sequence

$$0 \longrightarrow \underset{X_0}{S^{\oplus n}} \xrightarrow{d_0} \underset{X_1}{S^{\oplus n}} \xrightarrow{e} M \longrightarrow 0 \qquad (*)$$

Note that f acts trivially on M. Indeed, if $m = e(s) \in M$ we have $fm = e(fs) = ed_0 d_1(s) = 0$ since the sequence $(*)$ is exact. It follows that M is naturally an R-module.

The R-modules which arise in this way are special. Let $\mathbf{K}(R)$ be the subcategory of R-modules which can be put into a short exact sequence of S-modules of the form[1]

$$0 \longrightarrow S^{\oplus n} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

When S is a regular local ring $\mathbf{K}(R)$ is the category of *Cohen-Macaulay modules*. This is not how Cohen-Macaulay modules are usually defined. For more on Cohen-Macaulay modules see [Yos90], and in particular Chapter 7 for more on the relationship between Cohen-Macaulay modules and matrix factorisations[2].

R-modules to algebraic sets

Recall that the *annihilator* of an S-module M is the ideal

$$\operatorname{Ann}_S(M) = \{s \in S \mid sM = 0\}$$

[1]This is not standard notation.
[2]This is the context in which matrix factorisations were first introduced in [Eis80].

of S. For example, the annihilator of the S-module $R = S/(f)$ is the ideal (f).

Now let (X, d) be a matrix factorisation of $f \in S$ and M the associated Cohen-Macaulay module. We have shown that M is naturally an R-module, so $fM = 0$, or in other words we have the following inclusion of ideals of S:

$$(f) \subseteq \mathrm{Ann}_S(M)$$

We now focus on the case that $S = k[x_1, \cdots, x_m]$ for some commutative ring k. This allows us to look at the zero sets of each of these ideals. This gives

$$\mathbf{V}(f) \supseteq \mathbf{V}(\mathrm{Ann}_S(M))$$

and so we have associated to a matrix factorisation (X, d) of f an algebraic subset the zero set of f. If we assume k is Noetherian (for example if $k = \mathbb{C}$ or $k = \mathbb{R}$) all ideals of S are finitely generated, so say $\mathrm{Ann}_S(M) = (g_1, \cdots, g_r)$. This gives a concrete description of $\mathbf{V}(\mathrm{Ann}_S(M))$. In summary:

$$\underset{\text{M.F. of } f}{(X, d)} \xrightarrow{\text{coker}} \underset{\mathbf{K}(R)}{M} \xrightarrow{\mathbf{V}(\mathrm{Ann}_S(-))} \underset{\text{Algebraic subset of } \mathbf{V}(f)}{\mathbf{V}(g_1, \cdots, g_n)}$$

Examples

Let $S = \mathbb{R}[x, y]$ and $f = x^2 - y^2$. Consider the following matrix factorisations of f:

$$\mathbb{R}[x, y] \xrightarrow{x^2-y^2} \mathbb{R}[x, y] \xrightarrow{1} \mathbb{R}[x, y] \tag{1}$$

$$\mathbb{R}[x, y] \xrightarrow{1} \mathbb{R}[x, y] \xrightarrow{x^2-y^2} \mathbb{R}[x, y] \tag{2}$$

$$\mathbb{R}[x, y]^2 \xrightarrow{P} \mathbb{R}[x, y]^2 \xrightarrow{Q} \mathbb{R}[x, y]^2 \tag{3}$$

$$\mathbb{R}[x, y] \xrightarrow{x+y} \mathbb{R}[x, y] \xrightarrow{x-y} \mathbb{R}[x, y] \tag{4}$$

where $P = \left(\begin{smallmatrix} x & y \\ y & x \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} x & -y \\ -y & x \end{smallmatrix}\right)$.

Matrix Factorisation	Module	Annihilator	Algebraic Subset
(1)	$\mathbb{R}[x, y]/(x^2 - y^2)$	$(x^2 - y^2)$	$\mathbf{V}(x^2 - y^2)$
(2)	0	$\mathbb{R}[x, y]$	$\emptyset$
(3)	$\mathbb{R}[x, y]^2/((x, y)) + ((y, x))$		
(4)	$\mathbb{R}[x, y]/(x + y)$	$(x + y)$	$\mathbf{V}(x + y)$

Algebraic sets to back to matrix factorisations?

Given an algebraic set $\mathbf{V}(I)$, where $I \subseteq S$ is an ideal, it is tempting to consider the R-module S/I. Unfortunately this module is not always in $\mathbf{K}(R)$ and I am currently not certain how to reliably associate a module in $\mathbf{K}(R)$ to $\mathbf{V}(I)$.

Suppose we find such a module M in $\mathcal{K}(R)$ such that $\mathrm{Ann}_S(M) = I$. From its free resolution

$$0 \longrightarrow S^{\oplus n} \xrightarrow{p} S^{\oplus n} \xrightarrow{e} M \longrightarrow 0$$

we can produce a matrix factorisation as follows. Given $s \in S^{\oplus n}$ we have that $fs \in \ker(e) = \mathrm{im}(p)$ since f acts trivially on M. Since the sequence is exact there is a unique $t \in S^{\oplus n}$ such that $fs = p(t)$. We define $q : S^{\oplus n} \to S^{\oplus n}$ as $q(s) = t$.

Lemma 3. *The following*

$$S^{\oplus n} \xrightarrow{p} S^{\oplus n} \xrightarrow{q} S^{\oplus n}$$

is a matrix factorisation of f.

Proof. Clearly $pq = f \cdot 1$, and using that p is a monomorphism one can see $qp = f \cdot 1$.

It remains to show that q is S-linear. For $s, s' \in S^{\oplus n}$ we have $p(q(s+s')) = fs + fs' = p(q(s)+q(s'))$, and since p is a monomorphism this gives $q(s+s') = q(s)+q(s')$. Likewise $q(as) = aq(s)$ for $a \in S$. □

This association sending a matrix factorisation (X, d) to the module $\mathrm{coker}(d_0)$ is functorial and we have shown it is essentially surjective. However, it does not induce an equivalence of categories. The issue is as follows. Consider a free resolution of a module M in $\mathbf{K}(R)$:

$$0 \longrightarrow S^{\oplus n} \xrightarrow{p} S^{\oplus n} \xrightarrow{e} M \longrightarrow 0$$

We can construct more free resolutions of M by adding free summands to the first two terms like so:

$$0 \longrightarrow S^{\oplus n} \oplus S \xrightarrow{p \oplus 1} S^{\oplus n} \oplus S \xrightarrow{e \oplus 0} M \longrightarrow 0$$

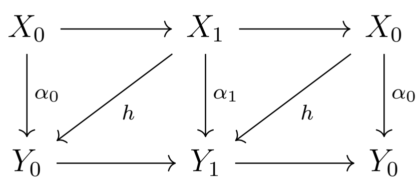
The matrix factorisations associated to each of these sequences are clearly not isomorphic. In other words, our functor identifies non-isomorphic matrix factorisations.

We can solve this issue by using a category of matrix factorisations with different morphisms, which we define in the next section. It turns out that this will be equivalent not to $\mathbf{K}(R)$, but to the quotient category $\mathbf{K}(R)/\{R\}$. The proof uses the same ideas discussed above and is given, in the case that S is a regular local ring (i.e. $\mathbf{K}(R)$ is the category of Cohen-Macaulay modules, but this makes little difference to the proof), in [Yos90, Chapter 7].

3 The homotopy category of matrix factorisations

Homotopy equivalent linear factorisations is defined analogously to complexes. We work with linear factorisations of $f \in S$, for S a commutative ring.

Definition 4. A *homotopy* of a morphism $\alpha : (X, d_X) \to (Y, d_Y)$ of linear factorisations is an odd S-linear map $h : X \to Y$ such that $\alpha = d_Y h + h d_X$. Diagrammatically:



where the triangles do not commute, but rather sum to α_i. We say two morphisms $\alpha, \beta : (X, d_X) \to (Y, d_Y)$ are *homotopic* if there exists a homotopy of $\alpha - \beta$.

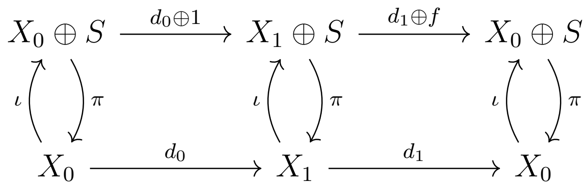
The homotopy category of linear factorisations is obtained by identifying homotopic morphisms. If two linear factorisations (X, d_X) and (Y, d_Y) are equivalent in the homotopy category then we say they are *homotopy equivalent*. This is the case if and only if there exist morphisms $\alpha : (X, d_X) \to (Y, d_Y)$ and $\beta : (Y, d_Y) \to (X, d_X)$ such that $\alpha\beta$ is homotopic to 1_Y and $\beta\alpha$ is homotopic to 1_X.

We denote the homotopy category of matrix factorisations of $f \in S$ by $\mathrm{HMF}(S, f)$, and the subcategory of matrix factorisations which are homotopic to a matrix factorisation of finite rank by $\mathrm{hmf}(S, f)$.

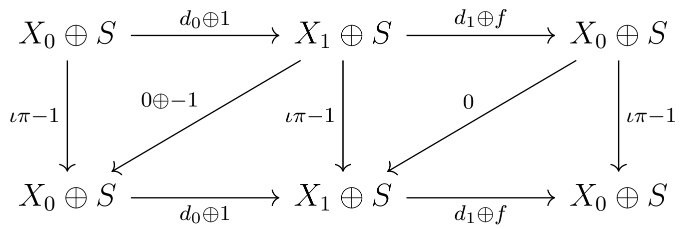
Example. Let (X, d) be a linear factorisation. Then the linear factorisation

$$X_0 \oplus S \xrightarrow{d_0 \oplus 1} X_1 \oplus S \xrightarrow{d_1 \oplus f} X_0 \oplus S$$

is always homotopy equivalent to (X, d), but not in general isomorphic to (X, d). Consider the morphisms



where ι and π are the inclusion and projection maps. We have $\pi\iota = 1$, so it remains to show that $\iota\pi$ is homotopic to 1. A homotopy is



In the following lemma we identify a finite rank matrix factorisation with a pair of matrices by choosing generators for the $S^{\oplus n}$. Call a finite dimensional matrix factorisation (P, Q) *reduced* if it is equal to $(1, f)$, or if both P and Q have no unit entries.

Lemma 5. *Every finite rank matrix factorisation is homotopy equivalent to a reduced matrix factorisation.*

Proof. See previous example. □

Theorem 6 ([Orl09]). *Let k be an algebraically closed field and let $f \in k[|\boldsymbol{x}|]$. Then $\mathrm{hmf}(k[|\boldsymbol{x}|], f)$ is the zero category if and only if the ring $R = k[|\boldsymbol{x}|]/(f)$ is regular.*

References

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