The Łojasiewicz exponent

We begin by stating definitions. ~~Throug~~ In the following we refer to Feehan (2019) (who at times is stating results from other work).

Let $k = \mathbb{R}$ or $\mathbb{C}$, & consider k^d.

<u>Theorem</u> (Łojasiewicz gradient inequality): Let $U \subseteq k^d$ be open and $f : U \to k$ be analytic. Let $x^* \in U$ be a critical point: $\nabla f(x^*) = 0$. For x^*, there exist constants $C > 0$, $\sigma \in (0,1]$ and $\theta \in [\frac{1}{2}, 1)$ such that

$$\|\nabla f(x)\| \geq C |f(x) - f(x^*)|^\theta \quad \text{for all } x \in B_\sigma(x^*) \qquad (1)$$

where $B_\sigma(x^*) \subseteq U$ is a σ-ball centred at x^*.

<u>Definition</u> The <u>Łojasiewicz exponent</u> of a differentiable function $f : U \to k$ is the smallest $\theta \geq 0$ such that there exist C, σ so that (1) holds, or more precisely:

$$\hat{\theta} = \inf \{ \theta \in [0, \infty) \mid \exists C > 0, \sigma \in (0,1] \text{ such that (1) holds} \}$$

We focus on the case of analytic functions, where the Łojasiewicz inequality ~~guarant~~ always exists & is greater than or equal to $\frac{1}{2}$.

- Feehan (2019) "Resolution of singularities & geometric proofs of the Łojasiewicz inequalities"

The Łojasiewicz gradient inequality has a corollary, sometimes ~~also~~ called the Łojasiewicz distance inequality. Various versions of this statement are possible, e.g. if the function is C^1 & the gradient inequality holds then so does the distance inequality. See Feehan Corollary 4, 5, 6. We follow the proof in Bierstone 1997.

Corollary. Let $U \subseteq \mathbb{R}^d$ be open and $f: U \to \mathbb{R}$ analytic. Let x^* be a critical point. Suppose that $\nabla f(x) = 0 \Rightarrow f(x) = 0$ for all $x \in B_r(x^*)$, ~~where~~ and that $f(x) \geq 0$. ~~Then~~ Let (C, σ, θ) be constants such that (1) is satisfied for x^* (Łojasiewicz gradient inequality).
Then:

$$f(x) \geq C' d(x, \overbrace{\cancel{f^{-1}(0)}}^{Z})^{\alpha} \quad \text{for all } x \in B_r(x)$$

where ~~$C' = (\theta C)^{1/\theta}$~~ $C' = ((1-\theta)C)^{\frac{1}{1-\theta}}$ and $\alpha = \frac{1}{1-\theta}$,
$Z = f^{-1}(0) \cap B_r(x^*)$

proof: Let $a \in B_r(x^*)$, st $f(a) \neq 0$. Suppose that $x(t)$ is a solution to

$$x'(t) = \frac{-\nabla f(x(t))}{\|\nabla f(x(t))\|}, \quad x(0) = a.$$

We define $Q(t) = f(x(t))$. Then

$$Q'(t) = \nabla f(x(t)) \cdot x'(t) = -\|\nabla (f(x(t))\|.$$

Hence for all t $Q'(t) < 0$, i.e. $Q(t)$ is decreasing.

Then

$$\frac{f(a)^{1-\theta}}{1-\theta} \geq \frac{f(x(0))^{1-\theta} - Q(t)^{1-\theta}}{1-\theta} = \frac{-1}{1-\theta}\int_0^t \frac{d}{dt} Q(t)^{1-\theta}\, dt.$$

$$= -\int_0^t Q(t)^{-\theta} Q'(t)\, dt$$

$$= \int_0^t f(x(t)) \cdot \|\nabla f(x(t))\|\, dt$$

$$\geq C\int_0^t dt = ct.$$

- Bierstone (1997) "Canonical desingularisation in characteristic zero by blowing up the maximum strata of a local invariant."

where in the last step we use Łojasiewicz inequality. Hence $x(t)$ converges to $f^{-1}(0)$ in finite time t_0. Since $\|x'(t)\| = 1$ we have $t_0 \geq d(a, Z)$. Rearranging gives

$$f(a) \geq ((1-\theta)c)^{1/1-\theta} \, d(a, Z)^{\frac{1}{1-\theta}} \qquad \square.$$

I ~~fel~~ feel like there are some frustrating details with this proof. What if $x(t)$ leaves $B_r(x^*)$? That seems to make the application of the gradient inequality invalid. ~~Likewise the dependence on a critical point is irritating~~. Lets put this issue aside (it will come up again).

Together these inequalities provide results about convergence of gradient flow:

Corollary. Let $f : U \to \mathbb{R}$ be an analytic function which is bounded below and let $f^* \in \mathbb{R}$ be the minimum (which we suppose is actually attained on U). Suppose $\nabla f(x) = 0 \Rightarrow f(x) = f^*$. Let θ be a value such that the Łojasiewicz inequality holds. ~~Then~~: Let $x(t)$ be a gradient flow of f starting sufficiently close to Z.

(1) If $\theta = \frac{1}{2}$ then the distance $d(x(t), Z)$ of $x(t)$ to Z decays to zero at least as quickly as Ke^{-ct} for some $K, c > 0$.

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Continuing Łojasiewicz exponents

~~Recall~~

Let $f: U \to \mathbb{R}$, $U \subseteq \mathbb{R}^d$ be an analytic function. Recall

Theorem (Łojasiewicz): Let $x^* \in U$ be a critical point of f, and suppose $\nabla f(x) = 0 \Rightarrow f(x) = f(x^*)$, and that $f(x) \geq f(x^*)$. Then exist constants $C', C > 0$, $\theta \in [\frac{1}{2}, 1)$ and a nbhd V of x^* such that

$$(1) \quad \|\nabla f(x)\| \geq C\,|f(x) - f(x^*)|^{\theta} \quad \text{for all } x \in V$$

$$(2) \quad |f(x) - f(x^*)| \geq C'\, d(x, Z)^{\alpha} \quad \text{— " —}$$

where $Z = f^{-1}(x^*)$ and $\alpha = \frac{1}{1-\theta}$.

The smallest value of θ satisfying (1) is the Łojasiewicz exponent of f at x^*.

One reason to ~~compu~~ care about the Łojasiewicz exponent is that it controls the convergence rate of gradient flow, with qualitatively different behavior at $\theta = \frac{1}{2}$ vs $\theta > \frac{1}{2}$.

Corollary Let f be as in the theorem and θ any value such that (1) & (2) hold. Let $x(t)$ be a gradient flow of f starting sufficiently close to x^*.

(1) If $\theta = \frac{1}{2}$ $\quad d(x(t), Z) \leq A e^{-at}$, $A, a > 0$.

(2) If $\theta > \frac{1}{2}$ $\quad d(x(t), Z) \leq A(t + B)^{-\frac{1-\theta}{2\theta - 1}}$ $\quad A, B > 0$.

The exponent $\frac{1-\theta}{2\theta - 1} \in (0, \infty)$ and is a decreasing function of θ. That is, when θ is larger convergence is slower.

proof: For simplicity suppose $x^* = 0$ & $f(x^*) = 0$. That is we consider the transformed function $f(x + x^*) - f(x^*)$.

~~When $\theta = \frac{1}{2}$ the Loj~~

We consider $Q(t) = f(x(t))$. Note that

$Q'(t) = \nabla f(x(t)) \cdot x'(t) = -\|\nabla f(x(t))\|^2$.

When $\theta = \frac{1}{2}$ the Łojasiewicz inequality says that

$$-Q'(t) \geq C\,Q(t)$$

Rearranging then integrating gives

$$-\int_0^t \frac{Q'(t)}{Q(t)}\,dt \geq Ct$$

$$\Rightarrow \quad -\log Q(t) \geq Ct + D$$

$$\Rightarrow \quad Q(t) \leq A e^{-at}, \quad A, a > 0.$$

The Łojasiewicz distance inequality gives

$$d(x(t), Z)^2 \leq Q(t)$$

and so

$$d(x(t), Z) \leq A' e^{-2at}$$

proving (1).

For $\theta > \frac{1}{2}$ we consider

$$Q(t)^{1-2\theta} = \int_0^t \frac{d}{dt} Q(t)^{1-2\theta}\,dt + Q(0)^{1-2\theta}$$

$$= \int_0^t (1-2\theta)\,Q'(t)\,Q(t)^{-2\theta}\,dt + D$$

$$Q(t)^{1-2\theta} = -\int_0^t (1-2\theta)\, \|\nabla f(x(t))\|^2 f(x(t))^{-2\theta}\, dt + D$$

$$Q(t)^{1-2\theta} = (2\theta - 1)\int_0^t \|\nabla f(x(t))\|^2 f(x(t))^{-2\theta}\, dt + D$$

Note that $(2\theta - 1) > 0$. Then:

$$Q(t)^{1-2\theta} \geq (2\theta - 1)\int_0^t c\, f(x(t))^{2\theta} f(x(t))^{-2\theta}\, dt + D$$

$$\geq A t + D$$

Rearranging (noting $1 - 2\theta < 0$) gives

$$Q(t) \leq A'(t + B)^{\frac{-1}{2\theta - 1}}$$

Applying the distance inequality gives

$$d(x(t), z) \leq A'(t + B)^{-\frac{1-\theta}{2\theta - 1}}$$

as required.

For functions in normal crossing form

Now suppose that f is in normal crossing form:

$$f(x) = a(x)\, x_1^{n_1} \cdot \ldots \cdot x_d^{n_d} \quad , \; n_i \in \{0, 1, 2, \ldots\}$$

where $a(0) \neq 0$. Also assume $d \geq 2$.

Theorem (Theorem 3 Feehan[1]): If f is as above and $\nabla f(0) = 0$ (i.e. at least one $n_i > 0$) then there exists a nbhd V of 0, $C > 0$ and such that

$$\| \nabla f(x) \| \geq C\, |f(x)|^{\theta}$$

where $\theta = 1 - \frac{1}{N}$, $N = \sum_{i=1}^{d} n_i$.

Note that if the hypothesis $\nabla f(0) = 0$ is satisfied we must have $N \geq 2$, and so $\theta \in [\frac{1}{2}, 1)$. Furthermore, an immediate corollary is that $\theta = \frac{1}{2}$ if and only if $f(x) = a(x) x_i^2$ or $f(x) = a(x) x_i x_j$ for some $i, j \in [d]$.

proof (Feehan[1] p12): ~~We have~~

$$\cancel{\| \nabla f(x) \| =}$$

~~Without~~ lo

By relabelling coordinates suppose $f(x) = a(x) \cdot \prod_{i=1}^{c} x_i^{n_i}$ for $c \leq d$. Then we have

$$\nabla f(x) = \sum_{i=1}^{d} \frac{\partial f}{\partial x_i}(x)\, e_i^*$$

$$\cancel{= \sum_{i=1}^{c} \left(x_i^{n_i} \frac{\partial a}{\partial x_i} + n_i x_i^{n_i - 1} a(x) \right) \prod_{j=1, j \neq i}^{c} x_j^{n_j}}$$

$$= \sum_{i=1}^{c} \left(x_i^{n_i} \frac{\partial a}{\partial x_i} + n_i x_i^{n_i - 1} a(x) \right) \cdot x_1^{n_1} \cdot \ldots \cdot \widehat{x_i^{n_i}} \cdot \ldots \cdot x_c^{n_c}\, e_i^*$$

$$+ \left(\prod_{i=1}^{c} x_i^{n_i} \right) \sum_{j=c+1}^{d} \frac{\partial a}{\partial x_j}\, e_j^*$$

[1] Feehan (2019) "Resolution of singularities and geometric proofs of the Łojasiewicz inequalities".

By slight abuse of notation:

$$\nabla f(x) = \prod_{i=1}^{c} x_i^{n_i} \cdot \left(\left(\sum_{i=1}^{c} \frac{\partial a}{\partial x_i} + n_i a(x) x_i^{-1} \right) e_i + \sum_{j=c+1}^{d} \frac{\partial a}{\partial x_j} e_j \right)$$

Thus, by just dropping "$\sum_{j=c+1}^{d} \frac{\partial a}{\partial x_j} e_j$" (note this is zero if $c=d$ & the following is an equality) we get

$$\| \nabla f(x) \|^2 \geq \left(\prod_{i=1}^{c} x_i^{2n_i} \right) \cdot \left(\sum_{i=1}^{c} \left(x_i \frac{\partial a}{\partial x_i} + n_i a(x) \right)^2 x_i^{-2} \right) \qquad (1)$$

Since $a(x)$ is C^1 & $c(0) \neq 0$ there is a ball $B_\sigma = B_\sigma(0)$ such that

$$\left| x_j \frac{\partial a}{\partial x_j} \right| \leq \frac{n_j}{2} |a(x)|, \quad \forall x \in B_\sigma, \; j = 1, \dots, c$$

(this just follows because $g(x) = \left| x_j \frac{\partial a}{\partial x_j} \right|$ and $h(x) = \frac{n_j}{2} |a(x)|$ are continuous and $g(0) < h(0)$). Thus,

$$n_j |a(x)| = \left| n_j a(x) + x_j \frac{\partial a}{\partial x_j} - x_j \frac{\partial a}{\partial x_j} \right|$$

$$\leq \left| n_j a(x) + x_j \frac{\partial a}{\partial x_j} \right| + \left| x_j \frac{\partial a}{\partial x_j} \right|$$

$$\leq \left| n_j a(x) + x_j \frac{\partial a}{\partial x_j} \right| + \frac{n_j}{2} \left| x_j \frac{\partial a}{\partial x_j} \right|$$

$$\Rightarrow \frac{n_j}{2} |a(x)| \leq \left| n_j a(x) + x_j \frac{\partial a}{\partial x_j} \right|$$

Since $n_j \geq 1$ for $j = 1, \dots, c$, from (1) we obtain

$$\| \nabla f(x) \|^2 \geq \frac{a(x)^2}{4} \prod_{i=1}^{c} x_i^{2n_i} \sum_{j=1}^{c} x_j^{-2} \qquad \forall x \in B_\sigma. \qquad (A)$$

~~Separately~~

Define

$$m = \inf_{x \in B_\sigma} |a(x)|, \qquad M = \sup_{x \in B_\sigma} |a(x)|$$

Note that since $a(x) \neq 0$ on $x \in \overline{B_\sigma}$ & $\overline{B_\sigma}$ is compact ~~$a(x)$~~ $m > 0$. Likewise $M < \infty$.

We first handle the case that $c=1$. In this case (A) becomes

$$\|\nabla f(x)\|^2 \geq \frac{a(x)^2}{4} x_1^{2(n_1-1)}$$

~~Noting $f(x) = a(x) x_1^{n_1}$ Taking square roots & using m gives~~

$$\cancel{\|\nabla f(x)\| \geq \frac{1}{2m} |x_1|^{n_1-1}}$$

~~Since $|f(x)| = a(x)|x_1|^{n_1}$~~

Thus

$$\|\nabla f(x)\| \geq \frac{1}{2}|a(x)|\,|x_1|^{n_1-1}$$

$$\geq \frac{1}{2}|a(x)|^{1/n_1}\left|a(x)x_1^{n_1}\right|^{1-\frac{1}{n_1}}$$

$$\geq \frac{1}{2} m^{1/n_1} |f(x)|^{1-\frac{1}{n_1}}$$

which proves the result when $c=1$.

Now suppose $c>1$ and set $N = \sum_{i=1}^{c} n_i$. Recall:

Lemma (Generalized Young Inequality): For $c \geq 2$, $a_j > 0$, $p_j > 0$ and $r > 0$ st $\frac{1}{r} = \sum_{i=1}^{c} \frac{1}{p_i}$ we have:

$$\left(\prod_{i=1}^{c} a_i\right)^r \leq r \sum_{i=1}^{c} \frac{a_i^{p_i}}{p_i} \quad (2)$$

Then we have that

$$\prod_{i=1}^{c} x_i^{\frac{-2n_i}{N}} \leq \frac{1}{N}\sum_{j=1}^{c} n_j x_j^{-2} \quad \cancel{(3)}$$

is true as an instance of (2) with $a_j = x_j^{\frac{-2n_j}{N}}$, $p_j = \frac{N}{n_j}$ and $r=1$. Setting $n = \max\{n_j\}$, we immediately get

$$\prod_{i=1}^{c} x_i^{\frac{-2n_i}{N}} \leq \frac{n}{N}\sum_{j=1}^{c} x_j^{-2} \quad (3)$$

From proof of the RHS of (A), applying (3) gives us

$$\prod_{i=1}^{c} x_i^{2n_i} \cdot \sum_{j=1}^{c} x_j^{-2} \geq \prod_{i=1}^{c} x_i^{2n_i} \cdot \prod_{i=1}^{c} x_i^{\frac{-2n_i}{N}} \cdot \frac{N}{n}$$
$$\geq \frac{N}{n} \prod_{i=1}^{c} x_i^{2n_i(1-\frac{1}{N})}$$

Then, from (A) + above we get

$$\|\nabla f(x)\| \geq \frac{1}{2}|a(x)| \cdot \sqrt{\frac{N}{n}} \left(\prod_{i=1}^{c}|x_i|^{n_i}\right)^{1-\frac{1}{N}}$$
$$\geq \frac{1}{2}|a(x)|^{1/N} \cdot \sqrt{\frac{N}{n}} \left(\left|a(x) \cdot \prod_{i=1}^{c} x_i^{n_i}\right|\right)^{1-\frac{1}{N}}$$
$$\geq \frac{1}{2} m^{1/N} \cdot \sqrt{\frac{N}{n}} \, |f(x)|^{1-\frac{1}{N}}$$

which completes the proof. $\square$.

We now consider a general analytic map f. ~~and~~ For simplicity suppose $f(0)=0$ and $\nabla f(0) = 0$. Let $g: \tilde{U} \to U$ be ~~som~~ a resolution of singularities of f in a neighborhood of the origin:

$$f \circ g(u) = a(u) \cdot \prod_{i=1}^{c} u_i^{n_i}$$

where $c \leq d$.

Pf (Łojasiewicz gradient inequality, Feehan p17): Let $N = \sum_{i=1}^{c} n_i$. Applying the previous theorem gives

$$\|(\nabla f \circ g)(u)\| \geq C\,|f \circ g(u)|^{1-\frac{1}{N}}, \quad \cancel{u \in \tilde{V} \subset \tilde{U}}$$

$u \in B_r \subset \tilde{U}$.

~~Let $x \in U$, and $u \in \tilde{U}$ st $x = g(u)$~~

Now g is an open map (?) so $g(B_r) = V$ is open. Let $x \in V$ and $u \in B_r$ be such that $x = g(u)$.

Using the chain rule

$$\|(\nabla f \circ g)(u)\| = \|\nabla f(g(u)) \cdot Jg(u)\| \leq \underbrace{\|Jg(u)\|}_{\leq M} \cdot \|\nabla f(x)\|$$

Combining these results gives

$$\cancel{\|(\nabla f \circ g)(u)\|}$$

$$\|\nabla f(x)\| \geq \ell' \, |f(x)|^{1-\frac{1}{N}}.$$