Blow-ups

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Algebraic geometry is the study of solutions to polynomial equations. This is an important area of study because the solutions to polynomial equations describe many familiar geometrical objects. For example

- $x^2 + y^2 + z^2 = 1$ is the 2-sphere,
- $x^2 + y^2 - z^2 = 1$ is the hyperboloid,
- $(x^2 + y^2 + 3)^2 = 8(x^2 + y^2)$ is the torus.

The point of this obvious statement is to convey the idea that algebraic geometry is more than an approach to commutative algebra which makes use of geometric analogies. Varieties – particular sets of solutions to polynomial equations, and the central object of study – are geometrical objects. Even when working over a field (or ring) which is not $\mathbb{R}$ or $\mathbb{C}$, certain definitions (the Zariski topolopgy) amount to assuming a minimal topological structure on our space.

1 Affine varieties

Without loss of generality we consider the *zero sets* of polynomial equations – the solutions of the equation given by setting a polynomial equal to zero. We now follow [1, Chapter 1]. Let k be a field[1]. We call k^n *affine n-space*[2].

Definition 1. An *affine algebraic set* is a subset of k^n which is the zero set of a collection of polynomials in $k[x_1, \cdots, x_n]$. That is, it is a set of the form

$$V(S) \coloneqq \{\boldsymbol{a} \in k^n : \forall f \in S \; f(\boldsymbol{a}) = 0\}$$

where $S \subseteq k[x_1, \cdots, x_n]$. An *affine variety* is an affine algebraic set which is not the union of two other non-empty affine algebraic sets, and a *quasi-affine variety* is an open subset of an affine variety (open in the subspace topology).

One can observe that:

- k^n is an affine variety, since $k^n = V(0)$.
- The set $V(S)$ depends only on the ideal generated by S, that is $V((S)) = V(S)$.
- Since k is Noetherian, so is $k[x_1, \cdots, x_n]$ (this is the Hilbert Basis Theorem) and so for any $S \subseteq k[x_1, \cdots, x_n]$ there is a finite $S' \subseteq k[x_1, \cdots, x_n]$ with $V(S) = V(S')$.

[1] In [1] it is assumed that k is algebraically closed, which allows more results to be proved in the classical setting.

[2] In [1] this is denoted $\mathbf{A}^n_k$.

Another important observation is that the affine algebraic sets form the closed sets of a topology on k^n. Indeed, $k^n = V(0)$, $\emptyset = V(k[x_1, \cdots, x_n])$, $\bigcap V(S_i) = V(\bigcup S_i)$ and if $I, J \subseteq k[x_1, \cdots, x_n]$ are ideals then $V(I) \cup V(J) = V(IJ)$. We call this topology the *Zariski topology* on k^n. The Zariski topologies are the weakest topologies in which $\{0\} \subseteq k$ is closed and polynomials are continuous, so declaring affine algebraic sets to be closed is equivalent to having the ambient assumption that k^n has a sufficiently nice topology to make the geometry 'work'[3]. Some facts to get a feel for the Zariski topology:

- Suppose $k = \mathbb{R}$ or $k = \mathbb{C}$. If a set is closed (resp. open) in the Zariski topology on k^n then it is closed (resp. open) in the usual topology on k^n. If a map is continuous in the Zariski topology then it is continuous in the usual topology.
- Points in k^n are closed.
- The Zariski topology is not Hausdorff.
- Every affine algebraic set has a unique expression as the finite union of distinct, non-nested affine varieties[4] [1, Proposition 1.5].
- Every open subset of a variety (i.e. the quasi-affine varieties) is dense.

Finally we can define the dimension of a variety as follows. If Y is a variety then the *dimension* of Y is the length m of the longest chain of distinct varieties

$$\{a\} = Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_m = Y$$

contained in Y. This can be understood recursively: a variety which is a point has dimension 0, and a variety has dimension n if it strictly contains a variety of dimension $n-1$, and every other variety it strictly contains has dimension at most $n-1$. This recursive property also holds for the dimension of vector spaces, and the dimension of manifolds (where submanifolds are defined in the 'adapted chart' sense, not as a subset which is also a manifold).

2 Singularities of varieties

Recall the regular value theorem from real differential geometry. Let $f : U \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be a differentiable function. We say that a point $y \in \mathbb{R}^m$ is a *regular value* of f if for every $a \in f^{-1}(y)$ the Jacobian $J(a) = \left[\frac{\partial f_i}{\partial x_j}(a)\right]$ at a has rank m (i.e. it is a surjective linear map). If $y \in \mathbb{R}^m$ is a regular value then $f^{-1}(y)$ is a submanifold of dimension $n - m$, or equivalently $\operatorname{rank} J(a) = \operatorname{codim} f^{-1}(y) = n - \dim f^{-1}(y)$.

Let $Y \subseteq k^n$ be an affine variety, and define

$$I(Y) = \{f \in k[x_1, \cdots, x_n] : \forall a \in Y \ f(a) = 0\}.$$

One can observe that $I(Y)$ is an ideal of $k[x_1, \cdots, x_n]$. We already have a good definition of the dimension of Y, so we define non-singular points of varieties as follows.

[3] Note that this assumption is weaker than k being a topological ring in which $\{0\}$ is closed. The latter assumes polynomials are continuous *with respect to the product topology on k^n*, whereas we only assume there is *some* topology on k^n making polynomials continuous. The Zariski topology on k^n is not the product topology. For example $\{(x, x) : x \in k\}$ is closed in the Zariski topology on k^2 but not in the product topology on k^2.

[4] This follows from the fact that the Zariski topology is *Noetherian* (every descending chain of closed sets stabilises) and is a general fact about Noetherian topologies. In a general topological space a closed set which is not the union of two other non-empty closed sets (eg a variety) is called *irreducible*.

Definition 2. Let $f_1, \cdots, f_k$ generate $I(Y)$. A point $a \in Y$ is *non-singular* if the formal Jacobian $\left[\frac{\partial f_i}{\partial x_j}(a)\right]$ has rank $n - \dim Y$. Otherwise $a \in Y$ is called a *singularity* of Y.

If Y is nonsingular at every point then it is called *non-singular variety*. When working over a field like $\mathbb{R}$ or $\mathbb{C}$ non-singular varieties are exactly the manifolds, and the singular points of a singular variety and the points which are not locally isomorphic to an open subset of $k^{\dim Y}$.

Note that if $Y = V(g_1, \cdots, g_m)$ then it is not necessarily true that rank $\left[\frac{\partial g_i}{\partial x_j}(a)\right] < n - \dim Y$ implies a is a singularity of Y, since the $g_1, \cdots, g_m$ may not generate $I(Y)$. For example consider $Y = \{x \in \mathbb{R} : g(x) = x^2 = 0\}$. The Jacobian $g''(0) = 0$, but $I(Y) = (x)$, which is not generated by x^2. This highlights an important difference between singularities of functions and singularities of sets (varieties). In Watanabe [2] we are mostly interested in singularities of functions, while blow-ups are an operations on varieties. The relationship between these two concepts is discussed more in Section 4.

3 Blow-ups

Blow-ups are procedure which removes singularities from a variety. Consider the variety Y defined by $y^2 = x^2(x+1)$ (Include a diagram) which has a singularity at $(x, y) = (0, 0)$. A way we might distinguish between the two different parts of Y intersecting $(0, 0)$ is to look at the slope at which they intersect the origin. The idea of the blow-up is to introduce additional variables which parameterise the lines passing through the origin.

Definition 3. We define *projective n-space* as $\mathbf{P}^n = k^{n+1} \setminus \{0\}/\sim$ where $p \sim q$ if and only if $p = \lambda q$ for some $\lambda \in k$.

We can understand $\mathbf{P}^n$ as the space of lines passing through the origin of k^{n+1}: the point $p = (p_1, \cdots, p_n) \in \mathbf{P}^n$ corresponds to the line $p_1 x_1 + \cdots + p_n x_n + p_{n+1} x_{n+1} = 0$. Polynomial functions $f \in k[x_1, \cdots, x_{n+1}]$ are well defined on $\mathbf{P}^{n+1}$ whenever every monomial in f has the same degree, since $f(\lambda p) = \lambda^{\deg f} f(p)$. We call such polynomial *homogeneous*. Just as for affine space, we define *projective algebraic sets* as subsets of $\mathbf{P}^n$ of the form

$$V(S) = \{p \in \mathbf{P}^n : \forall f \in S f(p) = 0\}$$

where $S \subseteq k[x_1, \cdots, x_{n+1}]$ is a set of homogeneous polynomials. As in the affine case a *projective variety* is defined as a projective algebraic set which is not the union of two other non-empty projective algebraic sets, and a *quasi-projective variety* is an open subset of a projective variety.

Lemma 4. *$\mathbf{P}^n$ is covered by $n + 1$ quasi-projective varieties, each isomorphic to k^n. In particular k^n is a quasi-projective variety.*

Proof. (sketch) We haven't defined morphisms of varieties (see [1, p. 15]), so I will just define the sets and provide the map which can be shown to be an isomorphism. For a full proof see [1, Proposition 2.2, Proposition 3.3]. Let $U_i = \{(p_1, \cdots, p_{n+1}) \in \mathbf{P}^n : p_i \neq 0\}$, noting that is is an open set. Then the maps

$$\varphi_i : U_i \to k^n \qquad \varphi_i(p_1, \cdots, p_{n+1}) = (p_1/p_i, \cdots, \hat{p}_i, \cdots p_n/p_i)$$

define isomorphisms of varieties, where $\hat{p}_i$ denotes that this element has been removed. It is at least easy to see they are bijections. □

We now follow [1, p. 28] in defining blow-ups.

Definition 5. Consider the product $k^n \times \mathbf{P}^{n-1}$ and polynomials $f \in k[x_1, \cdots x_n, y_1, \cdots, y_n]$ defining functions on $k^n \times \mathbf{P}^{n-1}$. The *blow-up* of k^n at $0 \in k^n$ is defined to be the zero set

$$X = \{x_i y_j = x_j y_i : i, j = 1, \cdots n\} \subseteq k^n \times \mathbf{P}^{n-1}.$$

Denote by $\varphi : X \to k^n$ the usual projection restricted to X.

If you are willing to believe that the product of quasi-projective varieties is a quasi-projective variety (this is [1, Exercise 3.6]) then note that $k^n \times \mathbf{P}^{n-1}$ is a quasi-projective variety and so X is closed in $k^n \times \mathbf{P}^{n-1}$.

Lemma 6. *$\varphi : X \setminus \varphi^{-1}(0) \to k^n$ is a bijection and $\varphi^{-1}(0) \simeq \mathbf{P}^n$.*

Proof. $\varphi^{-1}(0) = \{(0, \cdots, 0, y_1, \cdots, y_n) : (y_1, \cdots, y_n) \in \mathbf{P}^{n-1}\}$ so $\varphi^{-1}(0) \simeq \mathbf{P}^n$ is clear.

Now consider $(a_1, \cdots, a_n, p_1, \cdots, p_n) \in X \setminus \varphi^{-1}(0)$ and suppose $a_{i_0} \neq 0$. Then we have $p_j = (a_j / a_{i_0}) p_{i_0}$. Now $p_{i_0} \neq 0$, so we may take $p_{i_0} = a_{i_0}$ giving $(p_1, \cdots, p_n) = (a_1, \cdots, a_n)$. Therefore $(a_1, \cdots, a_n) \mapsto (a_1, \cdots, a_n, a_1, \cdots, a_n)$ is inverse to φ restricted to $X \setminus \varphi^{-1}(0)$. □

Definition 7. The blowup of a variety $Y \subseteq k^n$ is $\tilde{Y} = \overline{\varphi^{-1}(Y \setminus \{0\})}$.

We now present [1, Example 4.9.1]. Consider the variety Y given by the equation $y^2 = x^2(x+1)$. For $(p, q) \in \mathbf{P}^1$ we have

$$\varphi^{-1}(Y) = \{(x, y, p, q) \in k^2 \times \mathbf{P}^1 : y^2 = x^2(x+1), xp = yq\}$$

Let $U = \{(x, y, p, q) : q \neq 0\}$ and $V = \{(x, y, p, q) : p \neq 0\}$, noting that both are open. For $(x, y, p, q) \in U$ the map $(x, y, p, q) \mapsto (x, y, p/q)$ is an isomorphism $U \simeq k^3$. Setting $u = p/q$, under this isomorphism $\varphi^{-1}(Y) \cap U$ is given by the equations

$$\begin{aligned} \varphi^{-1}(Y) \cap U &\simeq \{(x, y, u) \in k^3 : y^2 = x^2(x+1),\ y = xu\} \\ &= \{(x, y, u) \in k^3 : x^2(u^2 - x - 1) = 0,\ y = xu\} \qquad (*) \\ &= \{(x, y, u) \in k^3 : x = y = 0\} \\ &\quad \cup \{(x, y, u) \in k^3 : x = u^2 - 1,\ y = xu\} \end{aligned}$$

Note that $\tilde{Y} \cap U = \{(x, y, u) \in k^3 : x = u^2 - 1\}$, and that this is a non-singular affine variety. The other irreducible component is $\varphi^{-1}(0) \cap U$, and is called the *exceptional set*. Note that $\tilde{Y}$ intersects $\varphi^{-1}(0) \cap U$ at $u = -1, 1$, which corresponds to the slope of each part of Y intersecting the origin in k^2. On V, setting $v = q/p$, we can similarly find

$$\begin{aligned} \varphi^{-1}(Y) \cap V \simeq \{(x, y, v) \in kc3 : x = y = 0\} \\ \cup \{(x, y, v) \in k^3 : 1 - yv^3 - v^2 = 0,\ x = yv\}. \end{aligned}$$

Watanabe [2] refers to these reparameterisations as *local (affine) coordinates for Y* (for example in [2, Example 3.14, Theorem 4.6]). Notice that the equation defining $(*)$ above above is in *normal crossing form* [2, Definition 2.8] and is one of the local coordinates in normal crossing form asserted to exist in [2, Theorem 3.6].

If Y has a singularity at 0 then the blowup has the effect of simplifying that singularity. Blow-ups at points away from zero can be found by first doing a linear change of coordinates.

4 Blow-ups in singular learning theory

The main tool from algebraic geometry used in singular learning theory is Hironaka's Resolution of Singularities (of [3]). In [2], Watanabe states three versions of this theorem: for analytic functions [2, Theorem 2.3], for varieties [2, Theorem 3.5], and for polynomial functions [2, Theorem 3.6]. In this section we will discuss the version for varieties and the version for polynomial functions.

The version of Hironaka's Theorem for varieties states that using finitely many blow-ups we can completely remove all singularities from a variety. The version of Hironaka's theorem for polynomial functions states that using finitely many blow-ups we can put the polynomial in *normal crossing form*, which is a form in which the singularities of the function are simple. This is defined as follows:

Definition 8. An analytic function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$ in *normal crossing form* at $(a_1, \cdots, a_n) \in U$ if

$$f(x) = a(x) \prod_{i=1}^{n} (x_i - a_i)^{k_i}$$

where $a(x)$ is an analytic function which is nowhere zero on an open neighbourhood U' of $(a_1, \cdots, a_n)$ and the k_i are non-negative integers.

Recall [1, Example 4.9.1] discussed in Section 4. We found a reparameterisation of the polynomial $f(x, y) = y^2 - x^2(x + 1)$ which put it in normal crossing form at 0 on line $(*)$. Since $f(x, y)$ generates the ideal of the variety $I(Y)$ the difference between the two versions of Hironaka's Theorem appears to be a decision about whether to consider the exceptional sets or not.

But as we discussed at the end of Section 2 a function being singular is not the same as its zero set being singular. Consider the the polynomial function $f(x, y) = (x+y)^2$. It has a critical point at $x = y = 0$, but the variety $\{(x, y) \in k^2 : (x+y)^2\} = \{(x, y) \in k^2 : x = -y\}$ is non-singular. Despite this, if we proceed with the blow up at $(x, y) = (0, 0)$ then the polynomial is parameterised as $(u + 1)^2 y^2 = 0$, $x = yu$ which is normal crossing.

References

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