Matrix factorisations as short exact sequences

Rohan Hitchcock

Abstract

In this note we define a category of matrix factorisations of $f \in S$ where S is a commutative ring and f is not a zero divisor. Traditionally this is done in the context of Cohen-Macaulay modules, however in this note we do so in a way which should only require knowledge of basic homological algebra. When f is a power series, the matrix factorisations of f are strongly connected to the singularities of the variety given by solutions to the equation $f = 0$ and as such matrix factorisations should be of interest wherever singularities arise.

1 Introduction

Matrix factorisations were first introduced in Eisenbud 1980 where it was shown that they are related to a certain class of Cohen-Macaulay modules. Later it was shown in Orlov 2009 that matrix factorisations are also related to the singularities of algebraic and analytic varieties. Matrix factorisations are often introduced in the context of Cohen-Macaulay modules, such as in Chapter 7 of Yoshino and Matsumura 1990, however this context is not necessary when one is interested in studying singularities. In this brief note we define a suitable category of matrix factorisations in a way which requires no knowledge of Cohen-Macaulay modules. The primary reference for this work is Chapter 7 of Yoshino and Matsumura 1990.

Throughout let S be a commutative ring and fix $f \in S$, where f is not a zero divisor. Let $R = S/(f)$.

2 A category of matrix factorisations

We begin by making a naïve definition of a matrix factorisation of f. Through the course of this note we will develop this into a more sophisticated object.

Definition 1. A *matrix factorisation* of f is a pair of square matrices (P, Q) with entries in S which satisfy

$$PQ = f \cdot I \qquad \text{and} \qquad QP = f \cdot I$$

where I denotes the appropriate identity matrix.

The primary tool we need to develop this idea is Theorem 1 below, which relates a matrix factorisation to a particular short exact sequence of S-modules. Using this we will define morphisms of matrix factorisations by considering morphisms of the corresponding short exact sequences. The majority of Section 2 is devoted to proving this theorem.

Theorem 1. *A matrix factorisation of f is equivalent to a short exact sequence of S-modules of the form*

$$0 \longrightarrow S^{\oplus n} \longrightarrow S^{\oplus n} \longrightarrow M \longrightarrow 0$$

where M is a R-module considered with its induced S-module structure.

Before proceeding with the proof of Theorem 1 we note the following lemma, which is where we require the hypothesis that f is not a zero divisor.

Lemma 2. *Let (P, Q) be a matrix factorisation of f. The S-module morphisms $p, q : S^{\oplus n} \to S^{\oplus n}$ induced by P and Q respectively are monomorphisms.*

Proof. Suppose $qs = qt$ for some morphisms $r, s : N \to S^{\oplus n}$. Then $pqs = pqt$ and so $f \cdot s = f \cdot t$. Since f is not a zero-divisor we obtain $s = t$. Similarly if $ps = pt$ then $s = t$. □

Proof of Theorem 1. Let (P, Q) be a matrix factorisation and $p, q : S^{\oplus n} \to S^{\oplus n}$ be the morphisms induced by P and Q respectively. Set $M = \operatorname{coker}(p)$. Then

$$0 \longrightarrow S^{\oplus n} \xrightarrow{p} S^{\oplus n} \xrightarrow{e} M \longrightarrow 0 \tag{1}$$

is a short exact sequence since p is a monomorphism by Lemma 2. It remains to show that M is naturally an R-module, or equivalently that f acts trivially on M. To see this, take $m \in M$ and let $s \in S^{\oplus n}$ be such that $e(s) = m$. Then $fm = fe(s) = e(fs) = e \circ p(q(m)) = 0$ since (1) is exact. Therefore $f \cdot M = 0$.

Now consider a short exact sequence of S-modules

$$0 \longrightarrow S^{\oplus n} \xrightarrow{p} S^{\oplus n} \xrightarrow{e} M \longrightarrow 0.$$

where M is an R-module, and note that $f \cdot M = 0$. We begin by defining a map $q : S^{\oplus n} \to S^{\oplus n}$ in the following way. Let $s \in S^{\oplus n}$ Then we have that $fs \in \ker(e) = \operatorname{im}(p)$ since $e(fs) = fe(s) = 0$. Since the sequence is exact we have a unique $t \in S^{\oplus n}$ such that $fs = p(t)$. We define $q(s) = t$.

Claim. *The map q is a morphism of S-modules.*

Proof. (of claim) Let $s, s' \in S^{\oplus n}$ and $a \in S$. First note that $p(q(s)) = fs$. Then we have $p(q(s + s')) = fs + fs' = p(q(s) + q(s'))$. Since p is a monomorphism this gives $q(s + s') = q(s) + q(s')$ as required. Similarly we find $q(as) = aq(s)$. □

Fix a basis for $S^{\oplus n}$ and let P be the matrix of p and Q the matrix of q. We claim that (P, Q) is a matrix factorisation of f. Indeed we have already shown that $PQ = f \cdot I$. Also since $pqp(s) = fp(s) = p(fs)$ we have $qp(s) = fs$ for all $s \in S^{\oplus n}$ and so $QP = f \cdot I$ as well.

Finally we note that as long as we keep the choice of basis for $S^{\oplus n}$ consistent throughout these two processes are inverse to each other. □

Therefore we can identify a matrix factorisation of f with a short exact sequence. We define a morphism of matrix factorisations to be a morphism of the corresponding short exact sequences and we denote the category of matrix factorisations by $\mathrm{mf}(f)$. Note that $\mathrm{mf}(f)$ is an abelian category.

We should note that with this definition the matrix factorisation (P, Q) is not isomorphic to the matrix factorisation (Q, P) in general. Clearly the distinction between (P, Q) and (Q, P) is artificial, which tells us that $\mathrm{mf}(f)$ is not yet the right notion of a category of matrix factorisations.

3 Reduced matrix factorisations

Not all matrix factorisations of f are interesting. For example let $P = \left(\begin{smallmatrix} f & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $Q = \left(\begin{smallmatrix} 1 & 0 \\ 0 & f \end{smallmatrix}\right)$. The corresponding short exact sequence is

$$0 \longrightarrow S \oplus S \xrightarrow{f \oplus \mathrm{id}} S \oplus S \xrightarrow{q \oplus 0} S\big/(f) \longrightarrow 0$$

where $q : S \to S\big/(f)$ is the usual quotient morphism. We can generate more uninteresting matrix factorisations by adding free summands to the first two terms of the sequence. In light of this we make the following definition.

Definition 2. A matrix factorisation (P, Q) is called *reduced* if the entries of P and Q are not units.

We now aim to define a category where every matrix factorisation is isomorphic to some reduced matrix factorisation. For this we need to use the idea of a *quotient category*, which we define below. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ a subcategory of $\mathcal{A}$

Definition 3. The *quotient category* $\mathcal{A}\big/\mathcal{B}$ is defined to be the category with the same objects as $\mathcal{A}$ and morphisms given by

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(A, B) = \mathrm{Hom}_{\mathcal{A}}(A, B)\big/\mathcal{B}(A, B)$$

where $\mathcal{B}(A, B)$ is the subgroup of $\mathrm{Hom}_{\mathcal{A}}(A, B)$ generated by morphisms which factor through direct sums of objects of $\mathcal{B}$. Note that every object in $\mathcal{B}$ is identified with the zero object in $\mathcal{A}\big/\mathcal{B}$.

The homotopy category of short exact sequences in $\mathcal{A}$ is an example of a quotient category. If A and B are short exact sequences in an abelian category $\mathcal{A}$ then we define morphisms in the homotopy category to be $\mathrm{Hom}(A, B) = \mathrm{Hom}_{\mathbf{Ses}}(A, B)/N$ where N is the subgroup of null-homotopic chain maps and $\mathbf{Ses} = \mathbf{Ses}(\mathcal{A})$ is the category of short exact sequences in $\mathcal{A}$. One can show that a chain map is null-homotopic if and only if it factors through some split short exact sequence. Therefore the homotopy category of short exact sequences is equal to $\mathbf{Ses}\big/\mathcal{S}$ where $\mathcal{S}$ is the subcategory of split short exact sequences.

In $\mathrm{mf}(f)$ the only split short exact sequences are direct sums of the matrix factorisation $(1, f)$. Unfortunately $\mathrm{mf}(f)\big/\{(1, f)\}$ does not make every matrix factorisation isomorphic to a reduced one, the main obstruction to this being that $(1, f)$ and $(f, 1)$ are not isomorphic. Instead we must consider the category $\mathrm{hmf}(f) := \mathrm{mf}(f)\big/\{(1, f), (f, 1)\}$.

Theorem 3. *In the category* $\mathrm{hmf}(f)$ *every matrix factorisation is either isomorphic to a reduced matrix factorisation or is the zero object.*

Proof. Let (P, Q) be a matrix factorisation of f. We proceed by induction on the number of unit entries of P. If P and Q contain no units or if they are 1×1 matrices then the result is immediate.

Suppose P contains a unit. Then by choosing an appropriate basis for $S^{\oplus n}$ we can assume P and Q are of the form

$$P = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & P' & \\ 0 & & & \end{array}\right) \qquad \text{and} \qquad Q = \left(\begin{array}{c|ccc} f & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & Q' & \\ 0 & & & \end{array}\right)$$

for some P' and Q'. This is because any matrix containing at least one unit can be put in the form of P above by using row and column operations. Therefore $(P,Q) \simeq (1,f) \oplus (P',Q') \simeq (P',Q')$. By the induction hypothesis (P',Q') is either the zero object or isomorphic to a reduced matrix factorisation.

Similarly if Q contains a unit we can assume P and Q are of the form

$$P = \left(\begin{array}{c|ccc} f & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & P' & \\ 0 & & & \end{array}\right) \qquad \text{and} \qquad Q = \left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \vdots & & Q' & \\ 0 & & & \end{array}\right)$$

and the result follows in the same manner. $\square$

4 Discussion

In this note we have developed a good notion of the category of matrix factorisations of f, which we call hmf(f). This category is often of interest whenever singularities arise, as the following result demonstrates.

Theorem 4 (Orlov 2009)**.** *Let k be an algebraically closed field and let $f \in k[|\boldsymbol{x}|]$. Then* $\mathrm{hmf}(f)$ *is the zero category if and only if the ring $R = k[|\boldsymbol{x}|]/(f)$ is regular.*

References

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