Idempotents in preadditive categories

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12 August 2022

In this note we give a brief introduction to idempotent morphisms and idempotent completion of categories, focusing on categories which are preadditive. Idempotent morphisms and idempotent completion is also covered in [Bor94, Section 6.5] (where "idempotent completion" is called "Cauchy completion") but no special attention is paid to preadditive categories.

1 Definitions and basic results

Let \mathcal{C} be a category, which for now we do not assume is preadditive.

Definition 1.1. An endomorphism $e: C \to C$ in \mathcal{C} is an *idempotent* if $e^2 = e$.

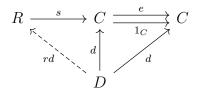
Consider a pair of morphisms $s: R \to C$, $r: C \to R$ such that $rs = \mathrm{id}_R$. Then e = sr is an idempotent.

Definition 1.2. We call an idempotent $e: C \to C$ split if there exist morphisms $s: R \to C$ and $r: C \to R$ such that e = sr and $rs = \mathrm{id}_R$. We call the category C idempotent complete if all idempotents split.

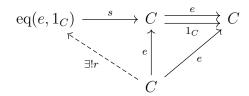
Lemma 1.3 ([Bor94, Proposition 6.5.4]). Let $e: C \to C$ be an idempotent. The following are equivalent:

- (1) $e = sr \text{ is split, where } s: R \to C \text{ and } r: C \to R.$
- (2) The equaliser eq $(e, 1_C)$ exists and is equal to (R, s).
- (3) The coequaliser $coeq(1_C, e)$ exists and is equal to (R, r).

Proof. Suppose e is split, so we have morphisms $s: R \to C$ and $r: C \to R$ such that $rs = \mathrm{id}_R$. We now prove that $\mathrm{eq}(e, 1_C) = (R, s)$ by showing the universal property is satisfied. We have es = s, and given another morphism $d: D \to C$ where ed = d, we have



where all three triangles commute. Indeed, setting n = rd we have sn = srd = ed = d. Moreover if $n': D \to R$ is another morphism which satisfies sn' = d we have sn = sn' and so rsn = rsn' = n = n'. This shows eq $(e, 1_C) = (R, s)$ and so $(1) \Longrightarrow (2)$. This also shows $(1) \Longrightarrow (3)$, since this is statement is equivalent to $(1) \Longrightarrow (2)$ holding in \mathcal{C}^{op} . Supposing (2), there exists $r: C \to eq(e, 1_C)$ such that



commutes. By applying the universal property to the morphism $s : eq(e, 1_C) \to C$ and appealing to uniqueness we have rs = 1, which proves (2) \Longrightarrow (1). By making use of \mathcal{C}^{op} this also shows (3) \Longrightarrow (1).

Lemma 1.4. If C is a preadditive category then the following are equivalent:

- (1) C idempotent complete.
- (2) All idempotents have a kernel.
- (3) All idempotents have a cokernel.

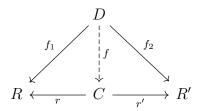
Proof. The equivalence (1) \iff (2) can be proved using Lemma 1.3 by observing that if $e: C \to C$ is an idempotent then so is $1_C - e$, and that $\operatorname{eq}(1_C - e, 1_C) = \ker(e)$. The equivalence (1) \iff (3) can be proved in the same way in the opposite category.

As a corollary note that any abelian category is idempotent complete. For an additive category, the property of "being idempotent complete" can be viewed as a weakening of "being abelian".

Lemma 1.5. Suppose C is preadditive. Let $e: C \to C$ be an idempotent such that the idempotents e and 1-e both split: e=sr and 1-e=s'r' where $s: R \to C$, $r: C \to R$, $s': R' \to C$ and $r': C \to R'$. Then $C \cong R \oplus R'$.

Proof. Since $rs = 1_R$ and $r's' = 1_{R'}$ we have that s and s' are monomorphisms, and r and r' are epimorphisms. Also note that rs' = 0 since rs'r' = r(1 - e) = 0 and r' is an epimorphism. Likewise r's = 0.

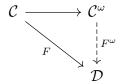
Suppose we have morphisms $f_1: D \to R$ and $f_2 = D \to R'$. Then we have



where $f = sf_1 + s'f_2$. Clearly both triangles in this diagram commute. Suppose $g: D \to C$ also makes both triangles in the diagram above commute. Then $r'g - r'sf_1 = f_2 = r's'f_2$ and, since r' is an epimorphism this gives g = f, so f is unique and hence $C = R \times R'$. A similar argument shows that C is also the coproduct of R and R', which proves the lemma.

2 Idempotent completion

Definition 2.1. The *idempotent completion* of \mathcal{C} is an idempotent complete category \mathcal{C}^{ω} together with a full and faithful functor $\mathcal{C} \to \mathcal{C}^{\omega}$ such that, given a functor $F: \mathcal{C} \to \mathcal{D}$ where \mathcal{D} is idempotent complete, there exists functor $F^{\omega}: \mathcal{C}^{\omega} \to \mathcal{D}$ such that



commutes, and moreover F^{ω} is unique up to isomorphism of functors.

Using the standard argument for objects defined via universal properties one can show that if \mathcal{C}^{ω} exists it is unique up to equivalence of categories. If \mathcal{C}^{ω} exists we can without loss of generality consider \mathcal{C} to be a full subcategory of \mathcal{C}^{ω} . In [Bor94, Proposition 6.5.9] it is proved that \mathcal{C}^{ω} exists when \mathcal{C} is small.

Our goal now is to prove that when \mathcal{C} is a subcategory of a preadditive, idempotent complete category \mathcal{A} , that \mathcal{C}^{ω} exists and is the full subcategory of \mathcal{A} of direct summands of objects of \mathcal{C} . Let A and B be objects of the same category. We say B is a retract of A if there exist morphisms

$$B \xrightarrow{s} A \xrightarrow{r} B$$

such that $rs = 1_B$.

Lemma 2.2. Let $\mathcal{C} \to \mathcal{D}$ be a fully faithful functor. Suppose \mathcal{D} is idempotent complete and that every object of \mathcal{D} is a retract of an object of \mathcal{C} . Then \mathcal{D} is the idempotent completion of \mathcal{C} .

Proof. Without loss of generality suppose \mathcal{C} is a full subcategory of \mathcal{D} . Let $F:\mathcal{C}\to\mathcal{E}$ be a functor to an idempotent complete category \mathcal{E} . We aim to construct a functor $\widetilde{F}:\mathcal{D}\to\mathcal{E}$ which fills in the diagram in Definition 2.1.

Let D be an object of \mathcal{D} and C an object of \mathcal{C} such that D is a retract of C, so we have

$$D \xrightarrow{s} C \xrightarrow{r} D$$

where $rs = 1_D$. In order to ensure \widetilde{F} is equal to F when restricted to objects of \mathcal{C} , if D happens to be an object of \mathcal{C} then choose C = D and $s = r = 1_D$. Consider the morphism $e = sr : C \to C$, which is an idempotent in \mathcal{C} . Since \mathcal{E} is idempotent complete F(e) splits and, using Lemma 1.3, we can define $\widetilde{F}(D) = \operatorname{eq}(F(e), 1_{F(C)})$. In the case that D is an object of \mathcal{C} choose the equaliser to be (F(C), 1). We denote the associated equaliser morphism in \mathcal{E} by $\sigma : \widetilde{F}(D) \to F(C)$. Also note that by the same argument as in Lemma 1.3 we have a morphism $\rho : F(C) \to \widetilde{F}(D)$ in \mathcal{E} such that $F(e) = \sigma \rho$ and $\rho \sigma = 1$.

Let $f: D_1 \to D_2$ be a morphism in \mathcal{D} . For i = 1, 2 let C_i be an object of \mathcal{C} such that D_i is a retract of C_i . Let $s_i: D_i \to C_i$ and $r_i: C_i \to D_i$ be the morphisms in \mathcal{D} in this retract and $e_i = s_i r_i$. Let $\rho_i: F(C_i) \to \widetilde{F}(D_i)$ and $\sigma_i: \widetilde{F}(D_i) \to F(C_i)$ be the morphisms in \mathcal{E} which split $F(e_i)$. Note that the composition $s_2 f r_1: C_1 \to C_2$ is a morphism in \mathcal{C}

and so in \mathcal{E} we have the diagram

$$\widetilde{F}(D_1) \xrightarrow{\sigma_1} F(C_1) \xrightarrow{F(e_1)} F(C_1)$$

$$\downarrow F(s_2 f r_1)$$

$$\widetilde{F}(D_2) \xrightarrow{\sigma_2} F(C_2) \xrightarrow{F(e_2)} F(C_2)$$

where \tilde{f} exists by the universal property of the equaliser. We define $\tilde{F}(f) = \tilde{f}\sigma_1$.

To see that this defines a functor, first consider the case when $D_1 = D_2$ and $f = 1_{D_1}$. Then \tilde{f} on the diagram above is a morphism such that $\sigma_1 \tilde{f} = F(e_1)$, so by uniqueness $\tilde{f} = \rho_1$. Therefore $\tilde{F}(1_{C_1}) = \rho_1 \sigma_1 = 1_{\tilde{F}(D_1)}$ as required. Consider another morphism $g: D_2 \to D_3$ in \mathcal{D} . We have the following diagram in \mathcal{E} :

$$\widetilde{F}(D_2) \xrightarrow{\sigma_2} F(C_2) \xrightarrow{F(e_2)} F(C_2)$$

$$\downarrow F(s_3 f r_2)$$

$$\widetilde{F}(D_3) \xrightarrow{\sigma_3} F(C_3) \xrightarrow{F(e_3)} F(C_3)$$

Let $\tilde{h}: F(C_1) \to \tilde{F}(D_3)$ be the unique morphism in \mathcal{E} satisfying $\sigma_3 \tilde{h} = F(s_3 gfr_1)$, so by definition we have $\widetilde{F}(qf) = \widetilde{h}\sigma_1$. Note that

$$\sigma_3(\tilde{g}\sigma_2\tilde{f}) = F(s_3gr_2)F(s_2fr_1) = F(s_3gfr_1)$$

and so $\tilde{h} = \tilde{g}\sigma_3\tilde{f}$ by uniqueness. Therefore $\tilde{F}(gf) = \tilde{F}(g)\tilde{F}(f)$ as required and hence we have shown that \widetilde{F} defines a functor and by construction $\widetilde{F}|_{\mathcal{C}} = F$.

For uniqueness, suppose we have two functors $\widetilde{F}_1, \widetilde{F}_2: \mathcal{D} \to \mathcal{E}$ such that $\widetilde{F}_1|_{\mathcal{C}} = \widetilde{F}_2|_{\mathcal{C}} =$ F. Let D be an object in \mathcal{D} and

$$D \xrightarrow{s} C \xrightarrow{r} D$$

be a retract with C an object of C. Then for i = 1, 2 we have the retract

$$\widetilde{F}_i(D) \xrightarrow{\widetilde{F}_i(s)} F(C) \xrightarrow{\widetilde{F}_i(r)} \widetilde{F}_i(D)$$

in \mathcal{E} . Noting that $\widetilde{F}_i(s)\widetilde{F}_i(r)=F(e)$, we have $(\widetilde{F}_i(D),\widetilde{F}_i(s))$ is the equaliser eq $(F(e),1_{F(C)})$ by Lemma 1.3. Therefore the functors are naturally isomorphic.

Corollary 2.3. Let C be a subcategory of an preadditive, idempotent complete category \mathcal{A} . Then \mathcal{C}^{ω} is the full subcategory of \mathcal{A} consisting of objects which are direct summands of objects of C.

Proof. Let \mathcal{D} be this subcategory. Clearly \mathcal{C} is a subcategory of \mathcal{D} , and every object of \mathcal{D} is a retract of some object of \mathcal{C} .

We now show that \mathcal{D} is idempotent complete. If $e: C \to C$ is an idempotent in \mathcal{D} then it splits in \mathcal{A} as e = sr where $s: R \to C$ and $r: C \to R$. By Lemma 1.5 R is a direct summand of C. Since C is a direct summand of an object of \mathcal{C} we have that R is a direct summand of the same object and hence R is in \mathcal{D} . Therefore the morphisms $s: R \to C$ and $r: C \to R$ are in \mathcal{D} and e splits in \mathcal{D} . Corollary 2.4. The idempotent completion of the category of free modules over a commutative ring R is the category of projective modules over R.

Proof. It is well-known that a module is projective if and only if it is a direct summand of a free module. \Box

3 Results used to define \mathcal{LG}_k

Let R be a commutative ring, $f \in R$. We quote the following result without proof.

Theorem 3.1 ([Nee01, Proposition 1.6.8]). Any triangulated category which admits all countable coproducts is idempotent complete.

Corollary 3.2. HMF(R, f) is idempotent complete.

Proof. HMF(R, f) admits all countable coproducts and the shift functor $X \mapsto X[1]$ induces a triangulated structure on HMF(R, f).

Corollary 3.3. $hmf(R, f)^{\omega}$ is the full subcategory of direct summands of objects of hmf(R, f).

Proof. See Corollary 2.3.

Lemma 3.4. Let C be a preadditive category with a zero object and C^{ω} its idempotent completion. A functor $C \times C \to C^{\omega}$ extends uniquely to a functor $C^{\omega} \times C^{\omega} \to C^{\omega}$.

Proof. We can embed \mathcal{C} into \mathcal{C}^{ω} via the functor $C \mapsto (C,0)$ or via the functor $C \mapsto (0,C)$. Using this we can show $(\mathcal{C} \times \mathcal{C})^{\omega}$ is $\mathcal{C}^{\omega} \times \mathcal{C}^{\omega}$ directly from the definition.

References

- [Bor94] Francis Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*. Vol. 50. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.
- [Nee01] Amnon Neeman. *Triangulated Categories*. Annals of Mathematical Studies 148. Princeton University Press, 2001.