

Idempotents in preadditive categories

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In this note we give a brief introduction to idempotent morphisms and idempotent completion of categories, focusing on categories which are preadditive. Idempotent morphisms and idempotent completion is also covered in [Bor94, Section 6.5] (where “idempotent completion” is called “Cauchy completion”) but no special attention is paid to preadditive categories.

1 Definitions and basic results

Let \mathcal{C} be a category, which for now we do not assume is preadditive.

Definition 1.1. An endomorphism $e : C \rightarrow C$ in \mathcal{C} is an *idempotent* if $e^2 = e$.

Consider a pair of morphisms $s : R \rightarrow C$, $r : C \rightarrow R$ such that $rs = \text{id}_R$. Then $e = sr$ is an idempotent.

Definition 1.2. We call an idempotent $e : C \rightarrow C$ *split* if there exist morphisms $s : R \rightarrow C$ and $r : C \rightarrow R$ such that $e = sr$ and $rs = \text{id}_R$. We call the category \mathcal{C} *idempotent complete* if all idempotents split.

Lemma 1.3 ([Bor94, Proposition 6.5.4]). *Let $e : C \rightarrow C$ be an idempotent. The following are equivalent:*

- (1) $e = sr$ is split, where $s : R \rightarrow C$ and $r : C \rightarrow R$.
- (2) The equaliser $\text{eq}(e, 1_C)$ exists and is equal to (R, s) .
- (3) The coequaliser $\text{coeq}(1_C, e)$ exists and is equal to (R, r) .

Proof. Suppose e is split, so we have morphisms $s : R \rightarrow C$ and $r : C \rightarrow R$ such that $rs = \text{id}_R$. We now prove that $\text{eq}(e, 1_C) = (R, s)$ by showing the universal property is satisfied. We have $es = s$, and given another morphism $d : D \rightarrow C$ where $ed = d$, we have

$$\begin{array}{ccccc}
 R & \xrightarrow{s} & C & \xrightarrow[e]{1_C} & C \\
 & \swarrow & \uparrow d & \searrow d & \\
 & & D & & \\
 & \swarrow rd & & & \\
 & & & &
 \end{array}$$

where all three triangles commute. Indeed, setting $n = rd$ we have $sn = srd = ed = d$. Moreover if $n' : D \rightarrow R$ is another morphism which satisfies $sn' = d$ we have $sn = sn'$ and so $rsn = rsn' = n = n'$. This shows $\text{eq}(e, 1_C) = (R, s)$ and so (1) \implies (2). This also shows (1) \implies (3), since this is statement is equivalent to (1) \implies (2) holding in \mathcal{C}^{op} .

Supposing (2), there exists $r : C \rightarrow \text{eq}(e, 1_C)$ such that

$$\begin{array}{ccccc}
 \text{eq}(e, 1_C) & \xrightarrow{s} & C & \xrightarrow[e]{1_C} & C \\
 & \swarrow \exists! r & \uparrow e & \searrow e & \\
 & & C & &
 \end{array}$$

commutes. By applying the universal property to the morphism $s : \text{eq}(e, 1_C) \rightarrow C$ and appealing to uniqueness we have $rs = 1$, which proves (2) \implies (1). By making use of \mathcal{C}^{op} this also shows (3) \implies (1). \square

Lemma 1.4. *If \mathcal{C} is a preadditive category then the following are equivalent:*

- (1) \mathcal{C} idempotent complete.
- (2) All idempotents have a kernel.
- (3) All idempotents have a cokernel.

Proof. The equivalence (1) \iff (2) can be proved using Lemma 1.3 by observing that if $e : C \rightarrow C$ is an idempotent then so is $1_C - e$, and that $\text{eq}(1_C - e, 1_C) = \ker(e)$. The equivalence (1) \iff (3) can be proved in the same way in the opposite category. \square

As a corollary note that any abelian category is idempotent complete. For an additive category, the property of “being idempotent complete” can be viewed as a weakening of “being abelian”.

Lemma 1.5. *Suppose \mathcal{C} is preadditive. Let $e : C \rightarrow C$ be an idempotent such that the idempotents e and $1 - e$ both split: $e = sr$ and $1 - e = s'r'$ where $s : R \rightarrow C$, $r : C \rightarrow R$, $s' : R' \rightarrow C$ and $r' : C \rightarrow R'$. Then $C \cong R \oplus R'$.*

Proof. Since $rs = 1_R$ and $r's' = 1_{R'}$ we have that s and s' are monomorphisms, and r and r' are epimorphisms. Also note that $rs' = 0$ since $rs'r' = r(1 - e) = 0$ and r' is an epimorphism. Likewise $r's = 0$.

Suppose we have morphisms $f_1 : D \rightarrow R$ and $f_2 : D \rightarrow R'$. Then we have

$$\begin{array}{ccccc}
 & & D & & \\
 & \swarrow f_1 & \vdots f & \searrow f_2 & \\
 R & \xleftarrow{r} & C & \xrightarrow{r'} & R'
 \end{array}$$

where $f = sf_1 + s'f_2$. Clearly both triangles in this diagram commute. Suppose $g : D \rightarrow C$ also makes both triangles in the diagram above commute. Then $r'g - r'sf_1 = f_2 = r's'f_2$ and, since r' is an epimorphism this gives $g = f$, so f is unique and hence $C = R \times R'$. A similar argument shows that C is also the coproduct of R and R' , which proves the lemma. \square

2 Idempotent completion

Definition 2.1. The *idempotent completion* of \mathcal{C} is an idempotent complete category \mathcal{C}^ω together with a full and faithful functor $\mathcal{C} \rightarrow \mathcal{C}^\omega$ such that, given a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ where \mathcal{D} is idempotent complete, there exists functor $F^\omega : \mathcal{C}^\omega \rightarrow \mathcal{D}$ such that

$$\begin{array}{ccc} \mathcal{C} & \longrightarrow & \mathcal{C}^\omega \\ & \searrow F & \downarrow F^\omega \\ & & \mathcal{D} \end{array}$$

commutes, and moreover F^ω is unique up to isomorphism of functors.

Using the standard argument for objects defined via universal properties one can show that if \mathcal{C}^ω exists it is unique up to equivalence of categories. If \mathcal{C}^ω exists we can without loss of generality consider \mathcal{C} to be a full subcategory of \mathcal{C}^ω . In [Bor94, Proposition 6.5.9] it is proved that \mathcal{C}^ω exists when \mathcal{C} is small.

Our goal now is to prove that when \mathcal{C} is a subcategory of a preadditive, idempotent complete category \mathcal{A} , that \mathcal{C}^ω exists and is the full subcategory of \mathcal{A} of direct summands of objects of \mathcal{C} . Let A and B be objects of the same category. We say B is a *retract* of A if there exist morphisms

$$B \xrightarrow{s} A \xrightarrow{r} B$$

such that $rs = 1_B$.

Lemma 2.2. *Let $\mathcal{C} \rightarrow \mathcal{D}$ be a fully faithful functor. Suppose \mathcal{D} is idempotent complete and that every object of \mathcal{D} is a retract of an object of \mathcal{C} . Then \mathcal{D} is the idempotent completion of \mathcal{C} .*

Proof. Without loss of generality suppose \mathcal{C} is a full subcategory of \mathcal{D} . Let $F : \mathcal{C} \rightarrow \mathcal{E}$ be a functor to an idempotent complete category \mathcal{E} . We aim to construct a functor $\tilde{F} : \mathcal{D} \rightarrow \mathcal{E}$ which fills in the diagram in Definition 2.1.

Let D be an object of \mathcal{D} and C an object of \mathcal{C} such that D is a retract of C , so we have

$$D \xrightarrow{s} C \xrightarrow{r} D$$

where $rs = 1_D$. In order to ensure \tilde{F} is equal to F when restricted to objects of \mathcal{C} , if D happens to be an object of \mathcal{C} then choose $C = D$ and $s = r = 1_D$. Consider the morphism $e = sr : C \rightarrow C$, which is an idempotent in \mathcal{C} . Since \mathcal{E} is idempotent complete $F(e)$ splits and, using Lemma 1.3, we can define $\tilde{F}(D) = \text{eq}(F(e), 1_{F(C)})$. In the case that D is an object of \mathcal{C} choose the equaliser to be $(F(C), 1)$. We denote the associated equaliser morphism in \mathcal{E} by $\sigma : \tilde{F}(D) \rightarrow F(C)$. Also note that by the same argument as in Lemma 1.3 we have a morphism $\rho : F(C) \rightarrow \tilde{F}(D)$ in \mathcal{E} such that $F(e) = \sigma\rho$ and $\rho\sigma = 1$.

Let $f : D_1 \rightarrow D_2$ be a morphism in \mathcal{D} . For $i = 1, 2$ let C_i be an object of \mathcal{C} such that D_i is a retract of C_i . Let $s_i : D_i \rightarrow C_i$ and $r_i : C_i \rightarrow D_i$ be the morphisms in \mathcal{D} in this retract and $e_i = s_i r_i$. Let $\rho_i : F(C_i) \rightarrow \tilde{F}(D_i)$ and $\sigma_i : \tilde{F}(D_i) \rightarrow F(C_i)$ be the morphisms in \mathcal{E} which split $F(e_i)$. Note that the composition $s_2 f r_1 : C_1 \rightarrow C_2$ is a morphism in \mathcal{C}

and so in \mathcal{E} we have the diagram

$$\begin{array}{ccccc}
\tilde{F}(D_1) & \xrightarrow{\sigma_1} & F(C_1) & \xrightarrow[\underset{1}{\rightrightarrows}]{F(e_1)} & F(C_1) \\
& \swarrow \exists! \tilde{f} & \downarrow F(s_2 f r_1) & & \\
\tilde{F}(D_2) & \xrightarrow{\sigma_2} & F(C_2) & \xrightarrow[\underset{1}{\rightrightarrows}]{F(e_2)} & F(C_2)
\end{array}$$

where \tilde{f} exists by the universal property of the equaliser. We define $\tilde{F}(f) = \tilde{f}\sigma_1$.

To see that this defines a functor, first consider the case when $D_1 = D_2$ and $f = 1_{D_1}$. Then \tilde{f} on the diagram above is a morphism such that $\sigma_1 \tilde{f} = F(e_1)$, so by uniqueness $\tilde{f} = \rho_1$. Therefore $\tilde{F}(1_{C_1}) = \rho_1 \sigma_1 = 1_{\tilde{F}(D_1)}$ as required.

Consider another morphism $g : D_2 \rightarrow D_3$ in \mathcal{D} . We have the following diagram in \mathcal{E} :

$$\begin{array}{ccccc}
\tilde{F}(D_2) & \xrightarrow{\sigma_2} & F(C_2) & \xrightarrow[\underset{1}{\rightrightarrows}]{F(e_2)} & F(C_2) \\
& \swarrow \exists! \tilde{g} & \downarrow F(s_3 g r_2) & & \\
\tilde{F}(D_3) & \xrightarrow{\sigma_3} & F(C_3) & \xrightarrow[\underset{1}{\rightrightarrows}]{F(e_3)} & F(C_3)
\end{array}$$

Let $\tilde{h} : F(C_1) \rightarrow \tilde{F}(D_3)$ be the unique morphism in \mathcal{E} satisfying $\sigma_3 \tilde{h} = F(s_3 g f r_1)$, so by definition we have $\tilde{F}(g f) = \tilde{h}\sigma_1$. Note that

$$\sigma_3(\tilde{g}\sigma_2\tilde{f}) = F(s_3 g r_2)F(s_2 f r_1) = F(s_3 g f r_1)$$

and so $\tilde{h} = \tilde{g}\sigma_2\tilde{f}$ by uniqueness. Therefore $\tilde{F}(g f) = \tilde{F}(g)\tilde{F}(f)$ as required and hence we have shown that \tilde{F} defines a functor and by construction $\tilde{F}|_{\mathcal{C}} = F$.

For uniqueness, suppose we have two functors $\tilde{F}_1, \tilde{F}_2 : \mathcal{D} \rightarrow \mathcal{E}$ such that $\tilde{F}_1|_{\mathcal{C}} = \tilde{F}_2|_{\mathcal{C}} = F$. Let D be an object in \mathcal{D} and

$$D \xrightarrow{s} C \xrightarrow{r} D$$

be a retract with C an object of \mathcal{C} . Then for $i = 1, 2$ we have the retract

$$\tilde{F}_i(D) \xrightarrow{\tilde{F}_i(s)} F(C) \xrightarrow{\tilde{F}_i(r)} \tilde{F}_i(D)$$

in \mathcal{E} . Noting that $\tilde{F}_i(s)\tilde{F}_i(r) = F(e)$, we have $(\tilde{F}_i(D), \tilde{F}_i(s))$ is the equaliser $\text{eq}(F(e), 1_{F(C)})$ by Lemma 1.3. Therefore the functors are naturally isomorphic. \square

Corollary 2.3. *Let \mathcal{C} be a subcategory of an preadditive, idempotent complete category \mathcal{A} . Then \mathcal{C}^ω is the full subcategory of \mathcal{A} consisting of objects which are direct summands of objects of \mathcal{C} .*

Proof. Let \mathcal{D} be this subcategory. Clearly \mathcal{C} is a subcategory of \mathcal{D} , and every object of \mathcal{D} is a retract of some object of \mathcal{C} .

We now show that \mathcal{D} is idempotent complete. If $e : C \rightarrow C$ is an idempotent in \mathcal{D} then it splits in \mathcal{A} as $e = sr$ where $s : R \rightarrow C$ and $r : C \rightarrow R$. By Lemma 1.5 R is a direct summand of C . Since C is a direct summand of an object of \mathcal{C} we have that R is a direct summand of the same object and hence R is in \mathcal{D} . Therefore the morphisms $s : R \rightarrow C$ and $r : C \rightarrow R$ are in \mathcal{D} and e splits in \mathcal{D} . \square

Corollary 2.4. *The idempotent completion of the category of free modules over a commutative ring R is the category of projective modules over R .*

Proof. It is well-known that a module is projective if and only if it is a direct summand of a free module. \square

3 Results used to define \mathcal{LG}_k

Let R be a commutative ring, $f \in R$. We quote the following result without proof.

Theorem 3.1 ([Nee01, Proposition 1.6.8]). *Any triangulated category which admits all countable coproducts is idempotent complete.*

Corollary 3.2. *$\text{HMF}(R, f)$ is idempotent complete.*

Proof. $\text{HMF}(R, f)$ admits all countable coproducts and the shift functor $X \mapsto X[1]$ induces a triangulated structure on $\text{HMF}(R, f)$. \square

Corollary 3.3. *$\text{hmf}(R, f)^\omega$ is the full subcategory of direct summands of objects of $\text{hmf}(R, f)$.*

Proof. See Corollary 2.3. \square

Lemma 3.4. *Let \mathcal{C} be a preadditive category with a zero object and \mathcal{C}^ω its idempotent completion. A functor $\mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}^\omega$ extends uniquely to a functor $\mathcal{C}^\omega \times \mathcal{C}^\omega \rightarrow \mathcal{C}^\omega$.*

Proof. We can embed \mathcal{C} into \mathcal{C}^ω via the functor $C \mapsto (C, 0)$ or via the functor $C \mapsto (0, C)$. Using this we can show $(\mathcal{C} \times \mathcal{C})^\omega$ is $\mathcal{C}^\omega \times \mathcal{C}^\omega$ directly from the definition. \square

References

- [Bor94] Francis Borceux. *Handbook of Categorical Algebra 1: Basic Category Theory*. Vol. 50. Encyclopedia of Mathematics and its Applications. Cambridge University Press, 1994.
- [Nee01] Amnon Neeman. *Triangulated Categories*. Annals of Mathematical Studies 148. Princeton University Press, 2001.