Bicategories

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In this note we define and motivate the notion of a bicategory, closely following [Bor94, Chapter 7].

1 2-categories

First observe that in any category $\mathcal{C}$ and A an object of $\mathcal{C}$, the identity morphism $\mathrm{id}_A : A \to A$ can be viewed as a morphism of sets $u_A : \{*\} \to \mathcal{C}(A, A)$ which identifies id_A (i.e. $u_A(*) = \mathrm{id}_A$). With this interpretation the identity axioms of the category ($\mathrm{id}_A\, f = f$ for all $f : B \to A$; $g \circ \mathrm{id}_A = g$ for all $g : A \to B$) can be expressed as the assertion that the following diagrams in the category of sets

$$\begin{tikzcd}
\{*\} \times \mathcal{C}(A,B) \arrow[d, "u_A \times \mathrm{id}"'] & \mathcal{C}(A,B) \arrow[l, "\cong"'] & \mathcal{C}(B,A) \arrow[r, "\cong"] & \mathcal{C}(B,A) \times \{*\} \arrow[d, "\mathrm{id} \times u_A"] \\
\mathcal{C}(A,A) \times \mathcal{C}(A,B) \arrow[ur, "c_{AAB}"'] & & & \mathcal{C}(B,A) \times \mathcal{C}(A,A) \arrow[ul, "c_{BAA}"]
\end{tikzcd} \tag{1}$$

commute for all pairs of objects A and B in $\mathcal{C}$, where $c_{XYZ} : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z)$ is composition. The associativity axiom can also be expressed as the assertion that the diagram of sets

$$\begin{tikzcd}
\mathcal{C}(A,B) \times \mathcal{C}(B,C) \times \mathcal{C}(C,D) \arrow[r, "\mathrm{id} \times c_{BCD}"] \arrow[d, "c_{ABC} \times \mathrm{id}"] & \mathcal{C}(A,B) \times \mathcal{C}(B,D) \arrow[d, "c_{ABD}"'] \\
\mathcal{C}(A,C) \times \mathcal{C}(C,D) \arrow[r, "c_{ACD}"'] & \mathcal{C}(A,D)
\end{tikzcd} \tag{2}$$

commutes for all objects A, B, C and D of $\mathcal{C}$.

Informally, a 2-category is a category in which we also have higher order morphisms between morphisms of objects, and everything "works". To be more precise:

Definition 1.1. A *2-category* $\mathcal{C}$ is a category in which $\mathcal{C}(A, B)$ is a category for all objects A and B, and:

(1) Composition $\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ is a functor.

(2) The map $u_A : \{*\} \to \mathcal{C}(A, A)$ which identifies the identity id_A is a functor, where $\{*\}$ is regarded as the category with one object and one morphism.

(3) The diagrams at (1) and (2) above commute as diagrams of categories.

The objects of $\mathcal{C}(A,B)$ are called *1-morphisms* and the morphisms of $\mathcal{C}(A,B)$ are called *2-morphisms*.

Following [Bor94], we denote objects in a 2-category by capital letters A, B, $C\dots$, 1-morphisms by lower-case letters a, b, $c\dots$, and 2-morphisms by Greek letters α, β, $\gamma\dots$. A 1-morphism can be denoted with an arrow $A \to B$ as usual, and a 2-morphism can be denoted with an arrow $a \Rightarrow b$.

In a 2-category $\mathcal{C}$ there are two notions of composition of 2-morphisms: applying the composition functor $c_{ABC} : \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C)$ to a pair of 2-morphisms, and composing 2-morphisms within $\mathcal{C}(A,B)$. For the first case, consider the following situation:

$$\begin{tikzcd} A \arrow[r, shift left=2, "f"] \arrow[r, shift right=2, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \arrow[r, shift left=2, "\ell"] \arrow[r, shift right=2, "m"'] \arrow[r, phantom, "\Downarrow\beta"] & C \end{tikzcd}$$

In the category $\mathcal{C}(A,B) \times \mathcal{C}(B,C)$ we have objects (f,ℓ) and (g,m) and a morphism $(\alpha,\beta) : (f,\ell) \Rightarrow (g,m)$. Via the composition functor c_{ABC} we obtain a 2-morphism $c_{ABC} : \ell \circ f \Rightarrow m \circ g$. Such composition is sometimes called *horizontal composition* or *composition along objects* and is denoted $\beta * \alpha := c_{ABC}(\alpha,\beta)$. This is in contrast to *vertical composition* or *composition along morphisms*, where in the situation of

$$\begin{tikzcd} A \arrow[r, shift left=4, "f"] \arrow[r, "g" description] \arrow[r, shift right=4, "h"'] \arrow[r, phantom, shift left=2, "\Downarrow\psi"] \arrow[r, phantom, shift right=2, "\Downarrow\varphi"] & B \end{tikzcd}$$

we obtain a morphism $\varphi \odot \psi : f \Rightarrow h$. Functoriality of composition means that in the situation of

$$\begin{tikzcd} A \arrow[r, shift left=4, "f"] \arrow[r, "g" description] \arrow[r, shift right=4, "h"'] \arrow[r, phantom, shift left=2, "\Downarrow\psi"] \arrow[r, phantom, shift right=2, "\Downarrow\varphi"] & B \arrow[r, shift left=4, "\ell"] \arrow[r, "m" description] \arrow[r, shift right=4, "n"'] \arrow[r, phantom, shift left=2, "\Downarrow\alpha"] \arrow[r, phantom, shift right=2, "\Downarrow\beta"] & C \end{tikzcd}$$

we have

$$\begin{aligned} (\beta * \varphi) \odot (\alpha * \psi) &= c_{ABC}(\varphi,\beta) \odot c_{ABC}(\psi,\alpha) \\ &= c_{ABC}((\varphi,\beta) \odot (\psi,\alpha)) \\ &= c_{ABC}((\varphi \odot \psi, \beta \odot \alpha) \\ &= (\beta \odot \alpha) * (\varphi \odot \psi) \end{aligned}$$

The prototypical example of a 2-category one in which the objects are categories, the 1-morphisms are functors and the 2-morphisms are natural transformations. Many familiar concepts in this setting transfer naturally to a general 2-category. For example:

Definition 1.2. A pair of 1-morphisms $f : A \leftrightarrows B : g$ in a 2-category are *adjoint* if there exist 2-morphisms $\eta : \mathrm{id}_B \Rightarrow f \circ g$ and $\epsilon : g \circ f \Rightarrow \mathrm{id}_A$ such that the diagrams

$$\begin{tikzcd} f \arrow[r, "\eta * i_f"] \arrow[dr, "i_f"'] & f \circ g \circ f \arrow[d, "i_f * \epsilon"] \\ & f \end{tikzcd} \qquad \begin{tikzcd} g \arrow[r, "i_g * \eta"] \arrow[dr, "i_g"'] & g \circ f \circ g \arrow[d, "\epsilon * i_g"] \\ & g \end{tikzcd}$$

commute in $\mathcal{C}(B,A)$ and $\mathcal{C}(A,B)$ respectively. Here i_f and i_g denote the identity 2-morphism on f and g respectively.

2 Bicategories

Consider any diagram of 1-morphisms in a 2-category $\mathcal{C}$, for example a square:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "g"] \\ C \arrow[r, "j"'] & D \end{tikzcd} \tag{$\Box$}$$

To say that ($\Box$) commutes is to say that $g \circ f = j \circ i$, or equivalently that we have the identity 2-morphism $\mathrm{id} : g \circ f \Rightarrow j \circ i$. This may be indicated on ($\Box$) by filling in its face like so:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "i"'] & B \arrow[d, "g"] \arrow[dl, phantom, "\Swarrow \mathrm{id}"] \\ C \arrow[r, "j"'] & D \end{tikzcd}$$

Of course, in a 2-category we may have 2-morphisms which are not the identity. It is therefore natural to consider diagrams which do not commute, but in which the faces are filled in with 2-morphisms. Of particular interest in defining bicategories are diagrams in which the faces are filled with 2-isomorphisms.

Suppose we have a collection of objects together with some candidate 1-morphisms and 2-morphisms. We would like to define a 2-category using this data, but the diagrams at (1) and (2) above do not commute. If these diagrams can instead can be filled in with natural isomorphisms (2-morphisms in a category of categories) then this data describes a bicategory.

Definition 2.1. A *bicategory* $\mathcal{B}$ consists of the following data:

(1) A collection of objects.

(2) For every pair of objects A, B a category $\mathcal{B}(A, B)$ of 1-morphisms.

(3) For each object A a functor $u_A : \{*\} \to \mathcal{B}(A, A)$, where $\mathrm{id}_A := u_A(*)$.

(4) For all objects A, B, C a composition functor $c_{ABC} : \mathcal{B}(A, B) \times \mathcal{B}(B, C) \to \mathcal{B}(A, C)$.

(5) For all objects A, B, C, D we have a natural isomorphism α_{ABCD} called the *associator* satisfying:

$$\begin{tikzcd} \mathcal{C}(A,B) \times \mathcal{C}(B,C) \times \mathcal{C}(C,D) \arrow[r, "\mathrm{id} \times c_{BCD}"] \arrow[d, "c_{ABC} \times \mathrm{id}"'] & \mathcal{C}(A,B) \times \mathcal{C}(B,D) \arrow[d, "c_{ABD}"] \\ \mathcal{C}(A,C) \times \mathcal{C}(C,D) \arrow[r, "c_{ACD}"'] \arrow[ur, phantom, "\Nearrow \alpha_{ABCD}"] & \mathcal{C}(A,D) \end{tikzcd}$$

(6) For all objects A, B we have natural isomorphism λ_{AB} and ρ_{AB} called the *left* and *right unitors* respectively, which satisfy:

$$\begin{tikzcd} \{*\} \times \mathcal{C}(A,B) \arrow[d, "u_A \times \mathrm{id}"'] & \mathcal{C}(A,B) \arrow[l, "\cong"'] \\ \mathcal{C}(A,A) \times \mathcal{C}(A,B) \arrow[ur, "c_{AAB}"'] \arrow[u, phantom, "\Nwarrow \lambda_{AB}"] & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}(A,B) \arrow[r, "\cong"] & \mathcal{C}(A,B) \times \{*\} \arrow[d, "\mathrm{id} \times u_A"] \\ & \mathcal{C}(A,A) \times \mathcal{C}(A,B) \arrow[ul, "c_{AAB}"] \arrow[u, phantom, "\rho_{AB} \Nearrow"] \end{tikzcd}$$

This data is subject to two conditions, called the *coherence conditions*, which are:

- Given 1-morphisms

  $$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{j} E$$

  the following diagram in $\mathcal{B}(A, E)$ commutes

  $$\begin{tikzcd}
((j \circ h) \circ g) \circ f \arrow[r, "\alpha_{g,h,j} * i_f"] \arrow[d, "\alpha_{f,g,j\circ h}"] & (j \circ (h \circ g)) \circ f \arrow[r, "\alpha_{f,g\circ h,j}"] & j \circ ((h \circ g) \circ f) \arrow[d, "i_j * \alpha_{f,g,j}"] \\
(j \circ h) \circ (g \circ f) \arrow[rr, "\alpha_{g\circ f,h,j}"] & & j \circ (h \circ (g \circ f))
\end{tikzcd}$$

  where we have written $\alpha_{f,g,h}$ for $\alpha_{ABCD}(f, g, h)$ and so on.

- Given 1-morphisms $A \xrightarrow{f} B \xrightarrow{g} C$ the following diagram in $\mathcal{B}(A, C)$ commutes

  $$\begin{tikzcd}
(g \circ i_B) \circ f \arrow[rr, "\alpha_{f,1_B,g}"] \arrow[dr, "\rho_g * i_f"'] & & g \circ (i_B \circ f) \arrow[dl, "i_g * \lambda_f"] \\
& g \circ f &
\end{tikzcd}$$

  where $i_B = u_B(* \to *)$ and again we write $\alpha_{f,1_B,h}$ for $\alpha_{ABBC}(f, 1_B, g)$ and so on.

Note that in general a bicategory is not a category. Thanks to the coherence conditions the definition of a bicategory appears very clunky in comparison to a 2-category. However, since many mathematical objects are defined only up to isomorphism, the bicategory is a more 'natural' concept.

Example. We can define a bicategory in which the objects are rings, and for rings R and S the category of 1-morphisms is the category of R-S-bimodules. Composition is given by taking the tensor product of bimodules. If we wanted to define a 2-category along these lines we would need to somehow arrange for the tensor products $(A \otimes_R B) \otimes_S C$ and $A \otimes_R (B \otimes_S C)$ to be equal — rather than naturally isomorphic — for associativity of composition to hold.

References

[Bor94] Francis Borceux. *Handbook of categorical algebra 1: basic category theory*. Encyclopedia of mathematics and its applications v. 50. Cambridge [England] ; New York: Cambridge University Press, 1994. 345 pp. ISBN: 0-521-44178-1.